PERGAMON

International Journal of Engineering Science 39 (2001) 201–208

International Journal of Engineering Science

www.elsevier.com/locate/ijengsci

Onset of convection in a porous medium with internal heat source and variable gravity

Isom H. Herron *

Center for Nonlinear Studies, Los Alamos National Laboratory, MS-B258, Los Alamos, NM 87545, USA

Received 2 June 1999; received in revised form 10 January 2000; accepted 10 January 2000

(Communicated by I. STAKGOLD)

Abstract

The problem treated is that of convection in a fluid saturated porous layer, heated internally and where the gravitational field varies with distance through the layer. The boundaries are assumed to be solid. It is proved that the principle of exchange of stabilities holds as long as the gravity field and the integral of the heat sources both have the same sign. The proof is based on the idea of a positive operator, and uses the positivity property of the Green's function. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Convection; Porous; Medium

1. Introduction

Interesting questions have been raised concerning the nature of the onset of instability due to convection in a fluid saturated porous layer with internal heat source and variable gravity [15]. The usual means of showing that the instability would be stationary derives from techniques applicable to Rayleigh–Bénard convection when the temperature difference is uniform and there are no internal heat sources and gravity is constant [1,13]. However, the more general cases have mainly been investigated one at a time. It is the purpose of this note to show that, when Darcy's law applies and the Boussinesq approximation is made, as long as the gravity field and the integral of the heat sources both have the same sign, with solid boundary conditions, the principle of exchange of stabilities (PES) holds in the following sense: "The first unstable eigenvalue has

* Permanent address: Department of Mathematical Sciences, Renssselaer Polytechnic Institute, Troy, NY 12180, USA.

PII: S0020-7225(00)00032-X

imaginary part equal to zero" [2,3,10]. When this eigenvalue is simple it is usually concluded that the onset of convection is stationary rather than oscillatory.

The appropriate non-dimensional disturbance equations for the flow are ([7,15]) from Darcy's law

$$A\frac{\partial \mathbf{u}}{\partial t} = -\nabla p + H(z)R\theta\mathbf{e}_z - \mathbf{u} \tag{1.1}$$

continuity

$$\nabla \cdot \mathbf{u} = 0 \tag{1.2}$$

and heat transport

$$\Pr\left(\frac{\partial \theta}{\partial t} + \mathbf{u}\cdot\nabla\theta\right) = RN(z)w + \Delta\theta, \tag{1.3}$$

where $\mathbf{u} = (u, v, w)$ is the seepage velocity, θ the temperature, $\mathbf{e}_z = (0, 0, 1)$, A an acceleration coefficient, R^2 the Rayleigh number, Pr the Prandtl number, $N(z) = 1 + \delta q(z)$, where $q(z)$ is proportional to the integral of the heat source and $H(z) = 1 + \varepsilon h(z)$; the gravity $g(z)$ being defined by $g(z) = g[1 + \varepsilon h(z)]$, g constant, and ε being a scale for h.

The acceleration coefficient A in (1.1) is often taken to be zero [15]. On the other hand, a more complete theory would require that this scalar be replaced by a second-order tensor that "depends sensitively on the geometry of the porous medium" [12, p. 9]. Strictly speaking then, the acceleration coefficient can only be considered a scalar if the flow is unidirectional, as happens at the beginning of convection, the regime to be considered in this work.

The equations are assumed to hold in the layer $\Omega = \{(x, y, z) \mid -\infty < x, y < \infty,\ 0 < z < 1\}$, with boundary conditions

$$\theta = 0, \quad z = 0, 1; \quad w = 0, \quad z = 0, 1. \tag{1.4}$$

The possibility of a non-zero tangential velocity is allowed. Next, linearize the perturbed system (1.1)–(1.3) and assume disturbances to be periodic in x (period $2\pi/\alpha$) and y (period $2\pi/\beta$) with growth rate σ, of the form

$$\mathbf{u} = \mathrm{e}^{\sigma t + \mathrm{i}\alpha x + \mathrm{i}\beta y}\hat{\mathbf{u}}(z)$$

for the velocity components, with comparable representations for θ and p. Then take curlcurl of (1.1) to obtain

$$A\sigma(D^2 - \gamma^2)w = (-D^2 + \gamma^2)w - \gamma^2 RH(z)\theta, \tag{1.5}$$

$$\Pr\sigma\theta = (D^2 - \gamma^2)\theta + RN(z)w, \tag{1.6}$$

where $D = \mathrm{d}/\mathrm{d}z,\ \gamma^2 = \alpha^2 + \beta^2$, and the hats have been dropped. The solid boundary conditions are

$$w = \theta = 0, \quad z = 0, 1. \tag{1.7}$$

If either $\varepsilon = 0$ or $\delta = 0$, then $H(z) \equiv 1$ or $N(z) \equiv 1$, and it is possible to show without much difficulty that if $\sigma = \sigma_1 + \mathrm{i}\sigma_2$ is complex, and $\sigma_2 \neq 0$, then $\sigma_1 < 0$. There is one interesting special case, namely when $H(z)$ is a multiple of $N(z)$, where the system can easily be made symmetric and then σ is real. [16, p. 66]. Though PES may seem to be intuitively true, when $H(z)$ and $N(z)$ are independent variables and $H(z)N(z) \geqslant 0$, no matter the boundary conditions, no results of this type had been proved. It is the purpose of this article to address this lack. In the next section, the proposed technique is described. Then, in the succeeding section the proof is carried out.

2. The method of positive operators

The idea of the method of solution is based on the notion of a *positive* operator [4,8], a generalization of a positive matrix, that is, one with all of its entries positive. Such matrices have the property that they possess a single greatest positive eigenvalue, identical to the spectral radius. To apply the method, the resolvent of the linearized stability operator is analyzed. This resolvent is in the form of compositions of certain integral operators. When the Green's function kernels for these operators are all non-negative, the resulting operator is termed positive. The abstract theory is based on the Krein–Rutman theorem.

Theorem 1 [11]. *If a linear, compact operator L, leaving invariant a cone $\mathfrak{K}$, has a point of the spectrum different from zero, then it has a positive eigenvalue λ, not less in modulus than every other eigenvalue, and to this number corresponds at least one eigenvector $\phi \in \mathfrak{K}$ of the operator L ($L\phi = \lambda\phi$), and at least one eigenvector $\psi \in \mathfrak{K}^*$ of the operator L^*.*

For this problem the cone consists of the set of non-negative functions.

In the formulation (1.5) and (1.6) with the boundary conditions (1.7), it is possible to rewrite the equations in terms of certain operators

$$(A\sigma + 1)\tilde{M}w - \gamma^2 RH(z)\theta = 0, \tag{2.1}$$

$$-RN(z)w + \left(\tilde{M} + \Pr\sigma\right)\theta = 0, \tag{2.2}$$

where

$$\tilde{M}w = (-D^2 + \gamma^2)w := mw, \quad w \in \operatorname{dom}\tilde{M},$$
$$\tilde{M}\theta = m\theta, \quad \theta \in \operatorname{dom}\tilde{M}.$$

The domains are contained in $\mathfrak{H}$, where

$$\mathfrak{H} = L^2(0,1) = \left\{\phi \,\middle|\, \int_0^1 |\phi|^2 \,\mathrm{d}z\right\} < \infty$$

with scalar product

$$\langle \phi, \psi \rangle = \int_0^1 \phi(z)\bar{\psi}(z)\,\mathrm{d}z, \quad \phi, \psi \in \mathfrak{H}$$

and norm

$$\|\phi\| = \langle \phi, \phi \rangle^{1/2}.$$

The domains are given as follows:

$$\operatorname{dom} \tilde{M} = \{\phi \in \mathfrak{H} \mid \phi',\ m\phi \in \mathfrak{H},\ \phi(0) = \phi(1) = 0\}.$$

With this formulation, it is apparent why the system (2.1) and (2.2) can be rendered symmetric when $H(z)$ is a multiple of $N(z)$ [16, p. 66], but otherwise the system is in general non-symmetric. The following gives important properties of the operator $\tilde{M}$.

Remark 1. *$\tilde{M}$ is a selfadjoint, positive definite operator. Furthermore, $\Gamma(\sigma) = (\tilde{M} + \sigma)^{-1}$ exists for*

$$\sigma \notin \Sigma_\gamma = \left\{\sigma \in \mathbf{C} \mid \mathrm{Re}(\sigma) \leqslant -\gamma^2,\ \mathrm{Im}(\sigma) = 0\right\}$$

and $\|\Gamma(\sigma)\|^{-1} > |\sigma + \gamma^2|$, for $\mathrm{Re}(\sigma) > -\gamma^2$ [9, p. 272]. *Explicitly, $\Gamma(\sigma)$ is the integral operator such that for $f \in \mathfrak{H}$*

$$\Gamma(\sigma)f = (\tilde{M} + \sigma)^{-1}f = \int_0^1 g(z, \xi; \sigma) f(\xi)\,\mathrm{d}\xi,$$

where

$$g(z, \xi; \sigma) = \frac{\cosh[r(1 - |z - \xi|)] - \cosh[r(-1 + z + \xi)]}{2r \sinh r} \tag{2.3}$$

is the appropriate Green's function and

$$r = \sqrt{\gamma^2 + \sigma}$$

is the positive square root.

It is now possible to write the system as single equation in w. Eliminating θ from (2.2) it follows that

$$\theta = \left(\tilde{M} + \mathrm{Pr}\,\sigma\right)^{-1} RN(z)w = \Gamma(\mathrm{Pr}\,\sigma)RN(z)w.$$

Similarly, in (2.1),

$$w = (A\sigma + 1)^{-1}\tilde{M}^{-1}\gamma^2 RH(z)\theta$$

when $(A\sigma + 1)^{-1} \neq 0$. So substituting for θ,

$$w = R^2\gamma^2(A\sigma + 1)^{-1}\tilde{M}^{-1}H(z)\Gamma(\Pr\sigma)N(z)w.$$

In a more compact form this equation is written as

$$w = K(\sigma)w. \tag{2.4}$$

What is to be studied is what follows in the resolvent $[I - K(\sigma)]^{-1}$. Suppose $K(\sigma)$ depends analytically on σ in a certain right half of the complex plane. Furthermore,

$$[I - K(\sigma)]^{-1} = \{I - [I - K(\sigma_0)]^{-1}[K(\sigma) - K(\sigma_0)]\}^{-1}[I - K(\sigma_0)]^{-1}. \tag{2.5}$$

So, if for all real σ_0 greater than some a

(P1) $[I - K(\sigma_0)]^{-1}$ is positive,

(P2) $K(\sigma)$ has a power series about σ_0 in $(\sigma_0 - \sigma)$ with positive coefficients, i.e. $(-\mathrm{d}/\mathrm{d}\sigma)^n K(\sigma_0)$ is positive for all n, then the right-hand side of (2.5) has an expansion in $(\sigma_0 - \sigma)$ with positive coefficients. Moreover the methods of [17] and [14] apply, showing that "there exists a real eigenvalue $\sigma_1 \leqslant a$ such that the spectrum of $K(\sigma)$ lies in the set $\{\sigma \mid \mathrm{Re}(\sigma) \leqslant \sigma_1\}$". This is equivalent to PES.

To verify conditions **(P1)** and **(P2)**, the structure of the operator $\Gamma(\Pr\sigma)$ will be examined. It is assumed that the product of the functions $H(z)N(z) \geqslant 0$ on [0, 1].

3. The principle of exchange of stabilities

Condition **(P1)** is treated first. The operator $\tilde{M}^{-1} = \Gamma(0)$ is an integral operator whose Green's function $g(z, \xi; 0)$ is non-negative so $\tilde{M}^{-1}$ is a positive operator. Next observe that by the remark, $\Gamma(\Pr\sigma)$ is also an integral operator, but its Green's function kernel $g(z, \xi; \sigma)$ in (2.3) is the Laplace transform of the Green's function $G(z, \xi; t)$ for the initial-boundary value problem

$$\left(-\frac{\partial^2}{\partial z^2} + \gamma^2 + \Pr\frac{\partial}{\partial t}\right)G = \delta(z - \xi, t), \tag{3.1}$$

$$G(0, \xi; t) = G(1, \xi; t) = G(z, \xi; 0) = 0. \tag{3.2}$$

Using the method of images, or by direct calculation of the inverse Laplace transform we find

$$G(z, \xi; t) = \frac{\mathrm{e}^{-\gamma^2\tau}}{\sqrt{4\pi\tau}}\sum_{j=-\infty}^{\infty}\left\{\mathrm{e}^{-(z-\xi+2j)^2/4\tau} - \mathrm{e}^{-(z+\xi+2j)^2/4\tau}\right\}, \tag{3.3}$$

where $\tau = t/\Pr$. It follows easily that

$$G(z,\xi;t) \geqslant 0, \quad 0 \leqslant z,\xi \leqslant 1, \quad t > 0. \tag{3.4}$$

Since

$$g(z,\xi;\sigma) = \int_0^\infty e^{-\sigma t} G(z,\xi;t)\,dt \tag{3.5}$$

we see that

$$\left(-\frac{d}{d\sigma}\right)^n g(z,\xi;\sigma) = \int_0^\infty t^n e^{-\sigma t} G(z,\xi;t)\,dt \geqslant 0 \tag{3.6}$$

for all n and for all real $\sigma > -\gamma^2$.

It was shown in (3.3)–(3.6), that $\Gamma(\sigma) = (\tilde{M} + \sigma)^{-1}$ is a positive operator for all real $\sigma > -\gamma^2$, and that $\Gamma(\sigma)$ has a power series expansion about σ_0 in $(\sigma_0 - \sigma)$ with positive coefficients, i.e. $(-d/d\sigma)^n \Gamma(\sigma_0)$ is positive for all n and real $\sigma_0 > -\gamma^2$. Thus the expansion

$$\begin{aligned}\Gamma(\sigma) &= \Gamma(\sigma_0)[I - (\sigma_0 - \sigma)\Gamma(\sigma_0)]^{-1} \\ &= \Gamma(\sigma_0)[I + (\sigma_0 - \sigma)\Gamma(\sigma_0) + (\sigma_0 - \sigma)^2(\Gamma(\sigma_0))^2 + \cdots]\end{aligned} \tag{3.7}$$

is valid for $|\sigma_0 - \sigma|\|\Gamma(\sigma_0)\| < 1$. The coefficients are positive operators when $\sigma_0 > -\gamma^2$. The expansion (3.7) may be analytically continued to the whole half-plane $\mathrm{Re}(\sigma) > -\gamma^2$. It is possible now to complete the abstract analysis of the formulation (2.4) and obtain the desired result.

Theorem 2. *The principle of exchange of stabilities holds for* (2.1) and (2.2), *when the integrated internal heat sources $N(z)$ and variable gravity ratio $H(z)$ have the property that $H(z)N(z) \geqslant 0$ throughout the layer.*

Proof. The system (2.1) and (2.2) may be written as the single equation suggested by (2.4)

$$u = K(\sigma)u, \tag{3.8}$$

where

$$K(\sigma) = \gamma^2 R^2 (A\sigma + 1)^{-1} \tilde{M}^{-1} H(z) \Gamma(\Pr\sigma) N(z),$$

$$(A\sigma + 1)^{-1} \neq 0.$$

The resolvent is examined as defined in (2.5). It has been demonstrated that the original system (2.1) and (2.2), and the transformed system (3.8), have spectra that agree except on the set $\Sigma_{\gamma/\sqrt{\Pr}}$ which is a subset of the negative real half-line. We have shown that $\Gamma(\sigma)$ is a positive operator, $\tilde{M}^{-1} = \Gamma(0)$, and that $\Gamma(\Pr\sigma)$ has power series expansions for real $\sigma > -\gamma^2/\Pr$. For σ_0 real, $(A\sigma_0 + 1)^{-1} > 0$, for $\sigma_0 > -1/A$. To verify condition **(P2)** for all real $\sigma_0 > \max(-\gamma^2/\Pr, -1/A)$, again note that it is assumed that $H(z)N(z) \geqslant 0$, while γ^2 and R^2 are clearly positive. Therefore, by the product rule for differentiation, one concludes that $K(\sigma)$ in (3.8) satisfies condition **(P2)**.

It has been demonstrated that for σ real and sufficiently large, all of the terms in $K(\sigma)$ determine positive operators. Moreover, for σ real and sufficiently large, by (2.3), the kernel of the operator $\Gamma(\Pr\sigma)$ becomes arbitrarily small. Hence, $\|K(\sigma)\|$ will be less than 1. Then $[I - K(\sigma)]^{-1}$ has a convergent Neumann series and hence is positive. This is the content of condition **(P1)**. □

4. Concluding comments

By using Darcy's law, one avoids the difficulty occurring with no-slip boundary conditions which has always impeded solution of the problem. The solution of the Bénard convection problem with stress-free boundary conditions is possible with this method, although it is only when there is both an internal heat source and variable gravity effects acting together that these techniques are needed [16, p. 66]. It is a similar combination of conditions which made possible the solution of analogous problems for Görtler flow with a free surface [5], and for Langmuir circulations with stress free boundary conditions [6].

Acknowledgements

The author would like to thank the Center for Nonlinear Studies at Los Alamos National Laboratory for its hospitality during a visit, where this work was performed under the auspices of the US Department of Energy.

References

[1] S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, Clarendon Press, Oxford, 1961.
[2] S.H. Davis, On the principle of exchange of stabilities, Proc. R. Soc. A. 310 (1969) 341–358.
[3] G.P. Galdi, B. Straughann, Exchange of stabilities, symmetry and nonlinear stability, Arch. Rat. Mech. Anal. 89 (1985) 211–228.
[4] F.R. Gantmacher, Applications of the Theory of Matrices, Interscience, New York, 1959.
[5] I.H. Herron, Exchange of stabilities for Görtler flow, SIAM J. Appl. Math. 45 (1985) 775–779.
[6] I.H. Herron, A simple criterion for exchange of stabilities in a model of Langmuir circulations, Eur. J. Mech. B 15 (1996) 771–779.
[7] D.D. Joseph, Stability of Fluid Motions, vol. II, Springer, Berlin, 1976.
[8] S. Karlin, Positive operators, J. Math. Mech. 8 (1959) 907–937.
[9] T. Kato, Perturbation Theory for Linear Operators, second ed., Springer, Berlin, 1976.
[10] K. Kirchgässner, Bifurcation in nonlinear hydrodynamic stability, SIAM Rev. 17 (1975) 652–683.
[11] M.G. Krein, M.A. Rutman, Linear operators leaving invariant a cone in a Banach space, Uspehi Matem. Nauk 3 (1948) 3–95 (AMS Transl.; No. 26 (1950)).
[12] D.A. Nield, A. Bejan, Convection in Porous Media, second ed., Springer, New York, 1999.

[13] A. Pellew, R.V. Southwell, On maintained convection in a fluid heated from below, Proc. R. Soc. A 176 (1940) 312–343.
[14] P.H. Rabinowitz, Nonuniqueness of rectangular solutions of the Bénard problem, in: J. B. Keller, S. Antman (Eds.), Bifurcation Theory and Nonlinear Eigenvalue Problems, Benjamin, New York, 1969.
[15] S. Rionero, B. Straughan, Convection in a porous medium with internal heat source and variable gravity effects, Int. J. Engng. Sci. 28 (1990) 497–503.
[16] B. Straughan, The Energy Method, Stability, and Nonlinear Convection, Springer, New York, 1992.
[17] H. F. Weinberger, Exchange of stability in Couette flow, in: J.B. Keller, S. Antman (Eds.), Bifurcation Theory and Nonlinear Eigenvalue Problems, Benjamin, New York, 1969.