Random Walk, Diffusion, and Energy Decay

MAA-NAM DAVID BLACKWELL LECTURE

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Abstract. Random walk, a discrete process, has a striking connection to Brownian motion which is governed in the continuous limit by the diffusion equation. The energy is one property of a solution to this equation. The rate of energy decay will be examined as a means of characterizing solutions to the equation. The methods employed have important applications in continuum mechanics.

I truly want to thank the organizers for inviting me to give this talk. I sought to find a connection between my work and that of the great mathematical statistician David Blackwell. Hence, my choice of the title and subject. Another interesting connection is that, in 1947 M. Kac published a Chauvenet prize-winning article "Random Walk and the Theory of Brownian Motion" in the American Mathematical Monthly [4], based on a talk given the previous year at the annual meeting of the MAA!

Random walk and diffusion. For our discussion we will only consider the one-dimensional random walk, though extensions to higher dimensions are just as old. Imagine a "free" particle that can move along the $x-$axis in such a way it can move either to the right or left with probability $\frac{1}{2}$ a distance Δ, each step taking time τ. A probability density is assigned to this process. It is $P(n\Delta|m\Delta; s\tau) = P(n|m; s)$, which is the probability that the particle is at $m\Delta$ at time $s\tau$, if at the beginning it was at $n\Delta$. It is significant to notice that $P(n|m; s)$ is the probability that after s games of unbiased coin-tossing, the gain of a player is $\nu = m - n$. A deeper analysis of this is problem is a result for which David Blackwell is justly famous [1]. It was accomplished between the publication of the second and third editions of W. Feller's "An Introduction to Probability Theory and Its Applications", and is duly cited in the latter.

It is possible to write the probability as

$$\begin{aligned} P(n|m;s) &= \frac{s!}{(s+|\nu|)!(s-|\nu|)!2^{s-2}}, \text{ if } |\nu| \le s \text{ and } |\nu|+s \text{ is even,} \\ &= 0, \text{ otherwise.} \end{aligned}$$

We wish to approximate this result when s, $s-m$, and $s+m$ are large. One way of doing this is by use of Stirling's formula

$$n! \sim (2\pi n)^{1/2} n^n e^{-n}, \text{ as } n \to \infty. \qquad (\dagger)$$

As W. Feller points out: " ...the ratio of the two sides of (†) tends to unity as $n \to \infty$.

It is true that the difference between the two sides increases over all bounds, but it is the percentage error which really matters. It decreases steadily, and Stirling's approximation is remarkably accurate even for small n. In fact, the right side of (†) approximates 1! by 0.9221 and 2! by 1.919 and 5!=120 by 118.019. The percentage errors are 8 and 4 and 2 respectively [2, p. 52]."

Then approximating the probability distribution in this way, we obtain

$$P(n|m,s) \sim \left(\frac{2}{\pi s}\right)^{1/2} e^{-(m-n)^2/2s}.$$

This result was originally obtained by Einstein in 1905 in his theory of Brownian motion.

The striking connection between the discrete (random walk) and the continuous approaches is even deeper. The exact probability distribution satisfies the difference equation

$$P(n|m;s+1) = \frac{1}{2}P(n|m-1;s) + \frac{1}{2}P(n|m+1;s).$$

Suppose that Δ and τ approach 0 in such a way that

$$\frac{\Delta^2}{2\tau} = k,\ n\Delta \to x_0,\ s\tau = t$$

The probability density is then

$$P(x_0|x;t) = \frac{1}{2\sqrt{\pi k t}} e^{-(x-x_0)^2/4kt}.$$

Moreover, the difference equation itself goes over to the partial differential equation (PDE)

$$\frac{\partial P}{\partial t} = k\frac{\partial^2 P}{\partial x^2},$$

which is the diffusion equation, the basis of Einstein's theory. A detailed derivation is provided in the now-classic text by Lin and Segel [5].

Boundary Conditions. Consider a random walk which takes place on the interval $0 < x < L$, such that there lies a barrier at $x = L$, $L > 0$. This barrier absorbs any particle which touches it. An analysis of the difference equation provides the desired boundary condition. Likewise, if the barrier is assumed to a perfect reflector, another boundary condition results at $x = L$ [5].

To solve the diffusion equation then, we pose the problem as

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},\ \ 0 < x < L,\ t > 0,$$

$$u(x, 0) = \phi(x)$$

$$u(x, 0) = 0,\ u(0, L) = 0.$$

This problem is to determine the concentration $u(x, t)$ given the initial distribution $\phi(x)$ where the boundary conditions are absorbing at $x = 0$ and at $x = L$. Since there is absorption at $x = 0$ and $x = L$, one expects that eventually all particles will disappear. The probability distribution P above forces one to suspect this as well.

Energy decay. One may define the *energy* of a solution as

$$E(t) = \frac{1}{2}\int_0^L (u(x, t))^2 dx.$$

We see that

$$E'(t) = \frac{1}{2}\int_0^L 2uu_t dx = k\int_0^L uu_{xx} dx = -k\int_0^L u_x^2 dx, \qquad (\dagger\dagger)$$

after integration by parts, by virtue of the boundary conditions. This shows that the energy does decay, but does not give the rate of decay.

Now for any differentiable function $f(x)$ on $0 < x < L$, satisfying $f(0) = 0$, $f(x) = \int_0^x f'(\xi)d\xi$. Hence

$$f^2(x) = \left(\int_0^x f'(\xi)d\xi\right)^2 \leq x\int_0^x \left(f'(\xi)\right)^2 d\xi$$

from the Cauchy-Schwarz inequality for integrals. Since the integrand on the right is non-negative, it thus follows that

$$f^2(x) \leq x\int_0^L \left(f'(\xi)\right)^2 d\xi.$$

Performing the integral of each extreme of the last inequality leads to

$$\int_0^L f^2(x)dx \leq \int_0^L xdx\int_0^L \left(f'(\xi)\right)^2 d\xi = \frac{1}{2}L^2\int_0^L \left(f'(\xi)\right)^2 d\xi.$$

This is applied to (††) to give

$$E'(t) \leq -\frac{2k}{L^2}\int_0^L (u(x,t))^2 dx = -\frac{4k}{L^2}E(t).$$

This differential inequality has the ready solution

$$E(t) \leq E(0)e^{-4kt/L^2}.$$

Hence, $E(t) \to 0$ as $t \to \infty$. The rate of energy decay is exponential. There is a sharper rate of decay that may be obtained by invoking the Calculus of Variations, which gives that

$$E(t) \leq E(0)e^{-k\lambda_1 t},$$

where λ_1 is the smallest eigenvalue of the problem

$$y''(x) + \lambda y(x) = 0, \quad 0 < x < L$$

$$y(0) = 0, \quad y(L) = 0.$$

That is, $\lambda_1 = (\pi/L)^2$. The general fact that
if $f(0) = f(L) = 0$, *and* $f' \in \mathfrak{L}^2$, *then*

$$\frac{\pi^2}{L^2}\int_0^L f^2 dx \leq \int_0^L f'^2 dx,$$

is nowadays called a Poincaré inequality, but also has the names Rayleigh & Ritz associated with it.

Suppose instead, that reflecting boundary conditions hold at both ends, then the appropriate boundary conditions are

$$u_x(0,t) = u_x(L,t) = 0.$$

Now we cannot expect that decay of the solution to zero is the final state. This corresponds to the fact that $u = U_0 = const.$ satisfies the PDE and the boundary conditions. However, if the initial condition $u(x,0) = \phi(x)$ is also to be satisfied, the ambiguity of the constant can be removed. We suppose that the value U_0 is the final (equilibrium) state of the process, i.e. $\lim_{t\to\infty} u(x,t) = U_0$. If the diffusion equation is integrated from $x = 0$ to $x = L$ the result is

$$\frac{d}{dt}\int_0^L u(x,t)dx = 0.$$

Thus, the quantity $\int_0^L u(x,t)dx$ is conserved. In particular, its value at $t = 0$, $\int_0^L \phi(x)dx = \lim_{t\to\infty} \int_0^L u(x,t)dx = \int_0^L U_0 dx = U_0 L$. Hence

$$U_0 = \frac{1}{L}\int_0^L \phi(x)dx.$$

The question arises, at what rate does $u(x,t) \to U_0$? This question can be answered by setting $v(x,t) = u(x,t) - U_0$. Then

$$\begin{aligned}\frac{\partial v}{\partial t} &= k\frac{\partial^2 v}{\partial x^2},\ 0 < x < L,\ t > 0,\\ v(x,0) &= \phi(x) - U_0,\end{aligned}$$

$$v_x(0,t) = v_x(L,t) = 0.$$

Now, define the energy as

$$E(t) = \frac{1}{2}\int_0^L (v(x,t))^2 dx.$$

In a similar manner as before, the counterpart of (††) follows. This time, we make use of what is often called Wirtinger's inequality:

If $f'(0) = f'(L) = 0,\ f' \in \mathfrak{L}^2,$ *and* $\int_0^L f dx = 0,$ *then*

$$\frac{\pi^2}{L^2}\int_0^L f^2 dx \le \int_0^L f'^2 dx.$$

It turns out that since $\int_0^L v(x,t)dx = 0$, Wirtinger's inequality applies to v. That is,

$$\frac{\pi^2}{L^2}\int_0^L v^2 dx \le \int_0^L v_x^2 dx.$$

This gives

$$E(t) \le E(0)e^{-k\pi^2 t/L^2}.$$

Exponential decay is again predicted, but this time to the uniform condition of the average value of the initial state.

An extension of many of the ideas on energy decay to two dimensions and to a nonlinear diffusion equation are carried out in [3].

References

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[5] C.-C. Lin and L. A. Segel, Mathematics Applied to Deterministic Problems in the Natural Sciences, SIAM, Philadelphia, 1988.

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