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Instabilities in the Görtler Model for Wall Bounded Flows

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Abstract—The original Görtler model is analyzed with no-slip boundary conditions on the wall. These conditions have historically been the most difficult to treat. It is proved that the principle of exchange of stabilities holds, that is, the first unstable eigenvalue has imaginary part equal to zero. The techniques used involve factoring positive operators.

Keywords—Görtler, Vortices, Instability.

1. GÖRTLER INSTABILITY: CONCAVE WALL

This type of instability was described by Görtler for flow along a concave wall. The instability manifests itself in the form of streamwise oriented vortices with the fluid spiraling around an axis, of dimensions comparable to the local boundary layer thickness. In the last two decades, vast improvements have been made in the theory describing the occurrence of these vortices [1–5]. Nevertheless, the original formulation has many of the mathematical features of the improvements, and the analysis often involves assumptions, not completely justified mathematically, which began with the original formulation. For these reasons, the original model equations will be treated in this article.

The Görtler equations which describe small disturbances near a concave wall are typified by

$$\left(D^2 - a^2 - \sigma\right)\left(D^2 - a^2\right) v + a^2 \mu U u = 0, \tag{1}$$

$$\left(D^2 - a^2 - \sigma\right) u - U' v = 0, \tag{2}$$

with wall boundary conditions

$$v(0) = v'(0) = u(0) = 0, \tag{3}$$

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and decay at infinity

$$v, v', u \to 0, \qquad \text{as } \eta \to \infty. \tag{4}$$

This is the notation of Drazin and Reid [6], where $D = \frac{d}{d\eta}$, $U(\eta)$ is the velocity component of the basic flow along the wall, U' is its derivative, a is the wave number, μ is the small gap Taylor number (linearly proportional to the curvature), and σ is the eigenvalue, the temporal decay rate (or Laplace transform parameter). In problems defined on an infinite interval, the spectrum of the linearized operator may be continuous, completely devoid of eigenvalues. This is possible when a parameter such as μ is very small. However, when the parameter is sufficiently large, instability occurs in the form of a single eigenvalue. The principle of exchange of stabilities (PES) is always invoked when these equations are treated. It is simply: "the first unstable eigenvalue has imaginary part equal to zero". Some time ago [7], the first author treated (1),(2), but with free surface boundary conditions instead of (3). Later, another approach was introduced to handle the wall bounded case [8]. Unfortunately, that approach was not as direct as that of [7] and its correctness is questionable. The purpose of this letter is to provide a more direct demonstration. It is also an approach which works in the case of flow between coaxial rotating cylinders, Taylor-Couette flow [9], and for Rayleigh-Bénard convection with nonuniform properties [10]. This is an abstract formulation which makes use of the notion of a *positive operator*. The inverses of the differential operators containing the eigenvalue σ are resolvents expressible as certain integral operators with Green's function kernels. (The main difference in this case is that the equations are defined on a semi-infinite spatial interval and so the integral operators introduced are not compact. However, the requisite theory for this case has been developed [11].) When these operators are all of one (positive) sign, the resulting operator is termed positive. This will be demonstrated by forming a power series expansion, all of whose terms are positive operators. Then, use is made of the classical result that if a power series with all positive terms has a radius of convergence ρ, then ρ is a singular point of the function represented by the power series.

2. ABSTRACT FORMULATION

The differential equations may be written in operator form [8]

$$(M^*M + \sigma M)v + a^2\mu U u = 0, \tag{5}$$

$$U'v + \left(\tilde{M} + \sigma\right) u = 0, \tag{6}$$

where

$$(M^*M + \sigma M)v = \left(\left(-D^2 + a^2\right)^2 + \sigma\left(-D^2 + a^2\right)\right) v,$$

$$\left(\tilde{M} + \sigma\right) u = \left(-D^2 + a^2 + \sigma\right) u.$$

Thus, even though M, M^*, and $\tilde{M}$ operate formally in the same way, they have different domains by virtue of the different boundary conditions on u and v at $\eta = 0$. With the boundary conditions, it is straightforward to show the following.

(a) $\tilde{M} + \sigma$ has an inverse which is an integral operator given by

$$G(\sigma) = \left(\tilde{M} + \sigma\right)^{-1} f = \int_0^\infty g(\eta, \xi; \sigma) f(\xi)\, d\xi,$$

with

$$g(\eta, \xi; \sigma) = \frac{e^{-r|\eta - \xi|} - e^{-r(\eta + \xi)}}{2r}, \tag{7}$$

where $r = \sqrt{\sigma + a^2}$ is the positive square root. For the special value $\sigma = 0$, the operator $\tilde{M}^{-1}$ is an integral operator whose Green's function $g(\eta, \xi; 0)$ is clearly nonnegative so $\tilde{M}^{-1}$ is a positive operator. Next, observe that $(\tilde{M}+\sigma)^{-1}$ is also an integral operator, but its Green's function kernel $g(\eta,\xi;\sigma)$ in (7) is the Laplace transform of the Green's function $G(\eta,\xi;t)$ for the initial-boundary value problem

$$\left(-\frac{\partial^2}{\partial\eta^2}+a^2+\frac{\partial}{\partial t}\right)G=\delta(\eta-\xi)\delta(t), \tag{8}$$
$$G(\eta,\xi;0)=0, \tag{9}$$
$$G(0,\xi;t)=0, \qquad \lim_{\eta\to\infty}G(\eta,\xi;t)=0. \tag{10}$$

Using the method of images, or by direct calculation of the inverse Laplace transform, we find

$$G(\eta,\xi;t)=\frac{e^{-a^2t}}{\sqrt{4\pi t}}\left\{e^{-(\eta-\xi)^2/4t}-e^{-(\eta+\xi)^2/4t}\right\}. \tag{11}$$

It follows easily that

$$G(\eta,\xi;t)\geq 0, \qquad 0\leq\eta, \quad \xi<\infty, \quad t>0. \tag{12}$$

Since

$$g(\eta,\xi;\sigma)=\int_0^\infty e^{-\sigma t}G(\eta,\xi;t)\,dt, \tag{13}$$

we see that

$$\left(-\frac{\partial}{\partial\sigma}\right)^n g(\eta,\xi;\sigma)=\int_0^\infty t^n e^{-\sigma t}G(\eta,\xi;t)\,dt\geq 0, \tag{14}$$

for all n and for all real $\sigma>-a^2$.

(b) After slightly more computation, it can be shown that M^*M has an inverse, also an integral operator given by

$$(M^*M)^{-1}f=\int_0^\infty h(\eta,\xi;0)f(\xi)\,d\xi,$$

with

$$h(\eta,\xi;0)=\frac{(1+a\,|\eta-\xi|)\,e^{-a|\eta-\xi|}-\left(1+a(\eta+\xi)+2a^2\eta\xi\right)e^{-a(\eta+\xi)}}{4a^3}, \tag{15}$$

which is the Green's function of its corresponding differential operator. By a straightforward analysis, it follows that $h(\eta,\xi;0)\geq 0$ for $0\leq\eta,\xi<\infty$.

It is also possible to express as an integral operator $H(\sigma)=(M^*M+\sigma M)^{-1}$ for all σ such that $\sigma\notin\sum_a=\{\sigma\in\mathbf{C}\mid\mathrm{Re}\,(\sigma)\leq -a^2,\ \mathrm{Im}\,(\sigma)=0\}$. The appropriate Green's function is for $\eta<\xi$

$$\begin{aligned} h(\eta,\xi;\sigma)=\frac{1}{2\sigma r}\Bigg\{&\left[\frac{r}{2a}-\frac{1}{2}\right]\left(e^{(a\eta-r\xi)}+e^{a(\eta-\xi)}\right)-\left[\frac{r}{2a}+\frac{1}{2}\right]\left(e^{-(r\xi+a\eta)}+e^{-a(\eta+\xi)}\right)\\ &+e^{-r(\eta+\xi)}+e^{-(r\eta+a\xi)}\Bigg\},\\ h(\xi,\eta;\sigma)=h(\eta,\xi;\sigma),& \end{aligned} \tag{16}$$

for $\eta>\xi$, where $r=\sqrt{\sigma+a^2}$ is the positive square root. Unfortunately, although one suspects that the expression (16) has the positivity property mentioned in Part (a), it would still be necessary, in general, to analyze its inverse Laplace transform as in (11) in order to conclude the positivity of the time-dependent function from which it was derived. We were not able to conclusively provide that analysis. However, if the derivatives of the Laplace transform have the correct signs, then the positivity of the time-dependent function follows [12].

Consequently, to start with, the M-resolvent $H(\sigma)$ is expanded in a Neumann series

$$H(\sigma)=(M^*M)^{-1}\left[I+(-\sigma)B+(-\sigma)^2B^2+\cdots\right], \tag{17}$$

where

$$B = M(M^*M)^{-1} \tag{18}$$

is a bounded operator. The series is defined for $|\sigma|(\|B\|) < 1$. The idea is to analytically continue this series to the whole half-plane $\mathrm{Re}(\sigma) > -a^2$. If each of the terms in the series can be shown to have the positivity property, then $H(\sigma)$ and all its derivatives have it, which is enough to conclude PES, it will be shown.

In order to analyze $H(\sigma)$, use will also be made of the *generalized inverse* of M, called $M^\dagger$. It is the integral operator which is defined as

$$M^\dagger\phi = \int_0^\infty g^\dagger(\eta,\xi)\phi(\xi)\,d\xi.$$

The kernel $g^\dagger$ is the *generalized Green's function* [13] which satisfies the equation

$$\left(-\frac{\partial^2}{\partial\eta^2} + a^2\right) g^\dagger = \delta(\eta - \xi) - g_Q(\eta,\xi),$$

where $g_Q = 2ae^{-a(\eta+\xi)}$, with boundary conditions

$$g^\dagger(0,\xi) = \frac{\partial g^\dagger}{\partial\eta}(0,\xi) = 0, \quad g^\dagger, \frac{\partial g^\dagger}{\partial\xi} \to 0, \qquad \text{as } \eta \to \infty.$$

The kernel g_Q corresponds to the integral operator Q which operates as

$$Q\phi = \int_0^\infty g_Q(\eta,\xi)\phi(\xi)\,d\xi.$$

(The operator Q is the orthogonal projection onto the $\mathrm{nul}\,M^*$; it is one-dimensional and is spanned by $e^{-a\eta}$.) Then it follows that

$$MM^\dagger = I - Q, \tag{19}$$

while

$$M^\dagger M = I, \tag{20}$$

the identity. By taking adjoints, it follows that $M^*M^{\dagger *} = I$ and $M^{\dagger *}M^* = I - Q$. These identities lead to the following conclusion. Apply $\tilde{M}^{-1}$ to both sides of (19) giving

$$\tilde{M}^{-1}MM^\dagger = \tilde{M}^{-1}(I - Q).$$

It is a simple calculation using integration by parts to show that $\tilde{M}^{-1}M = I$, and to conclude that

$$M^\dagger = \tilde{M}^{-1}(I - Q). \tag{21}$$

By definition of the generalized inverse,

$$M^\dagger M^{\dagger *} = (M^*M)^\dagger = (M^*M)^{-1},$$

since M^*M is invertible, being positive (definite) self-adjoint. It also follows from (19) and (21) that

$$(M^*M)^{-1} = M^\dagger M^{\dagger *} = \tilde{M}^{-1}(I - Q)\tilde{M}^{-1}. \tag{22}$$

It was shown, by (15), that the first term $(M^*M)^{-1}$ in (17) is positive. Thus, the next term may be written

$$(M^*M)^{-1}B = M^\dagger M^{\dagger *}(I - Q)M^{\dagger *} = \tilde{M}^{-1}(I - Q)\tilde{M}^{-1}(I - Q)\tilde{M}^{-1}, \tag{23}$$

by (22), (18), and (21). It is possible to calculate the Green's function corresponding to the integral operator (22) as follows. The Green's function for B corresponds to the integral operator for (18) which is $(-\frac{\partial^2}{\partial\eta^2}+a^2)h(\eta,\xi;0)=g^{\dagger *}(\eta,\xi)$. Explicitly, this kernel is

$$g^{\dagger *}(\eta,\xi)=\frac{e^{-a|\eta-\xi|}-(1+2a\xi)e^{-a(\eta+\xi)}}{2a}.$$

The required expression for the kernel of (23) is

$$h_2(\eta,\xi)=\int_0^\infty h(\eta,\xi_1;0)g^{\dagger *}(\xi_1,\xi)\,d\xi_1 := h\circ g^{\dagger *}.$$

This is found to be

$$h_2(\eta,\xi)=\frac{\left(a^2(\eta-\xi)^2+3a\,|\eta-\xi|+3\right)e^{-a|\eta-\xi|}}{16a^5} - \frac{\left(2a^3\eta\xi(\eta+\xi)+a^2\left(\eta^2+4\eta\xi+\xi^2\right)+3a(\eta+\xi)+3\right)e^{-a(\eta+\xi)}}{16a^5}.$$

It is possible to show without difficulty, but with a modest amount of algebra, that $h_2(\eta,\xi)\geq 0$ for $0\leq\eta,\ \xi<\infty$. This signifies that (23) is a positive operator.

It has been shown that the first two terms in (17) are positive operators. That each of the succeeding terms are also positive follows by induction if one notices the process by which the first two terms acquire their positivity. The operator $(M^*M)^{-1}=\tilde{M}^{-1}(I-Q)\tilde{M}^{-1}$ in (22). The resulting nonnegative kernel $h(\eta,\xi;0)$ is given by the composition $g\circ(\delta-g_Q)\circ g$. Then the second term is $g\circ(\delta-g_Q)\circ h$, which is also nonnegative. The succeeding terms all have the same structure.

3. EXCHANGE OF STABILITIES

It is possible to write (5),(6) as a single equation in v, by eliminating u

$$v=K(\sigma)v, \tag{24}$$

where

$$K(\sigma)=a^2\mu(M^*M+\sigma M)^{-1}U\left(\tilde{M}+\sigma\right)^{-1}U'=a^2\mu H(\sigma)UG(\sigma)U'.$$

The criterion for exchange of stabilities to be employed was enunciated some time ago [14]. Consider the resolvent of K defined as $[I-K(\sigma)]^{-1}$. Furthermore,

$$[I-K(\sigma)]^{-1}=\left\{I-[I-K(\sigma_0)]^{-1}[K(\sigma)-K(\sigma_0)]\right\}^{-1}[I-K(\sigma_0)]^{-1}. \tag{25}$$

So, if for all real σ_0 greater than some α

(P1) $[I-K(\sigma_0)]^{-1}$ is positive,

(P2) $K(\sigma)$ has a power series about σ_0 in $(\sigma_0-\sigma)$ with positive coefficients, that is, $(-\frac{d}{d\sigma})^nK(\sigma_0)$ is positive for all n,

then the right side of (25) has an expansion in $(\sigma_0-\sigma)$ with positive coefficients. Moreover the methods of [14] apply, showing that "there exists a real eigenvalue $\sigma_1\leq\alpha$ such that the spectrum of $K(\sigma)$ lies in the set $\{\sigma\mid \mathrm{Re}\,(\sigma)\leq\sigma_1\}$". This is equivalent to the PES.

The resolvent is examined as defined in (25). It has been demonstrated that the original system (5),(6), and the transformed system (24), have spectra that agree except on the set Σ_a, which is a subset of the negative real half-line. We have shown in (17) that $H(\sigma)$ is a positive operator, and that $H(\sigma)$ and $G(\sigma)$ have power series expansions for real $\sigma_0>-a^2$.

To verify Condition (P2), again note that we suppose that $U(\eta)$ and $U'(\eta)$ do not change sign, while a^2 and μ are positive. Therefore, by the product rule for differentiation, one concludes that $K(\sigma)$ in (24) satisfies Condition (P2).

It has been demonstrated that all of the terms in $K(\sigma)$ determine positive operators. Moreover, for σ real and sufficiently large, by (16) and (7), the kernels h and g become arbitrarily small and thus the norms of the operators $H(\sigma)$ and $G(\sigma)$ become arbitrarily small. Hence, $\|K(\sigma)\|$ will be less than 1. Then $[I - K(\sigma)]^{-1}$ has a convergent Neumann series and hence is positive. This is the content of Condition (P1).

4. CONCLUDING COMMENTS

The principle of exchange of stabilities holds for the original Görtler model for flow along a concave wall. This principle has been assumed for most of the succeeding models of the flow. The physical appearance of vortices originally suggested the validity of the principle. However, it has long since been known that the Görtler model equations predict instability at zero wave number. For this reason, among others, better models were developed. It is hoped that the analysis just provided will contribute to further analytical resolution of the instability phenomenon.

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