Analysis and Applications, Vol. 2, No. 2 (2004) 145–159

ONSET OF INSTABILITY IN HYDROMAGNETIC COUETTE FLOW

ISOM H. HERRON
Department of Mathematical Sciences, Rensselaer Polytechnic Institute
Troy, NY 12180, USA
herroi@rpi.edu

Received 21 October 2002
Revised 10 July 2003

The stability of viscous flow between rotating cylinders in the presence of a constant axial magnetic field is considered. The boundary conditions for general conductivities are examined. It is proved that the Principle of Exchange of Stabilities holds at zero magnetic Prandtl number, for all Chandrasekhar numbers, when the cylinders rotate in the same direction, the circulation decreases outwards, and the cylinders have insulating walls. The result holds for both the finite gap and the narrow gap approximation.

Keywords: Couette; stability; hydromagnetic.

Mathematics Subject Classification 2000: Primary: 76E25; Secondary: 47B65, 34B27

1. Introduction

It is now fifty years since Chandrasekhar published his first influential work on the theory of hydromagnetic stability [1]. Not quite as well known as some of his other work, it is all summarized in the monograph he prepared a decade later [2], where he considered the effects of a constant axial magnetic field on the instability of Couette flow. Shortly thereafter, Roberts [16] recognized that the boundary conditions employed in [2], which describe two extremes of non-conducting and infinitely (perfectly) conducting surfaces can be generalized to finite conductivity. Moreover, the system of differential equations which result, in considering general disturbances is of tenth order instead of eight as considered by most workers, who take the magnetic Prandtl number to be zero. Nevertheless, in cylindrical coordinates, the eighth order system has not been completely resolved. Most approaches have been to use the small-gap approximation so that the resulting equations are in Cartesian coordinates instead of cylindrical; Chandrasekhar [1,2] took this approach, as did Chang and Sartory [3] and this was more recently the subject of a detailed study by Chen and Chang [4]. Incorporating nonlinear effects, significant work was done earlier by Tabeling [18], using bifurcation theory, and by Galdi [7], using energy theory. Both of these authors used the small gap approximation. Tabeling [18], assumed axisymmetric disturbances, and considered both insulating and conducting boundary conditions. He performed numerical calculations and compared the results with experiments. His results for the linearized problem were confirmed and amplified in [4]. Galdi [7], considered three-dimensional disturbances for conducting boundary conditions. His linearized results were also confirmed in [4].

The objective of the current work is to develop the Principle of Exchange of Stabilities (PES); this has been stated as "*all non-decaying disturbances are non-oscillatory in time*" [2, 6]. Alternatively, it can be stated as, "the first unstable eigenvalue of the linearized system has imaginary part equal to zero" [8, 19]. The non-dimensional parameter of interest is the Chandrasekhar number Q, which measures the relative effects of the magnetic field and conductivity of the fluid to its viscosity. Already, the work of Chen and Chang [4] has shown conclusively that for relative rotation rates μ, with co-rotating cylinders, $0 < \mu < 1$, for the case of *perfectly conducting* walls, in the small-gap approximation, oscillatory instability will generally occur if $Q > 6080$. It should be mentioned too, that they found that unstable asymmetric modes will prevail for approximately $510 < Q < 6080$. However, it is the case of *insulating walls* which this paper intends to address. Admittedly, the phenomena are more "tame" in the sense that none of the switching phenomena with increasing Q, from axisymmetric to asymmetric to oscillatory, evident with conducting walls were seen to occur. Nevertheless, the approach here is analytical, and shows the natural connection with the non-magnetic case $Q = 0$, for which the PES is known to hold [11]. It will be seen that only in the insulating case does the proof of the PES for all Q go through. This is because when the governing equations are set in operator notation, it is precisely this case for which certain operators commute. In the next section, the governing equations are introduced, and the relevant boundary conditions described. Then the narrow gap approximate equations are presented. In the succeeding section an abstract formulation is made which links both the finite and narrow gap equations. The method of proof using positive operators emerges and finally the proof of the PES is given.

2. Governing Equations

The governing equations are those of magnetohydrodynamics (MHD) [5, 17]. The basic state is that of Couette flow between infinitely long rotating cylinders, of radii $R_1 < R_2$, and corresponding rotation rates Ω_1 and Ω_2, with constant axial magnetic field $B_0\mathbf{k}$. Perturbations to the basic state are assumed to be small, so that quadratic terms are ignored, giving a linearized system for the velocity and magnetic field disturbances.

2.1. *Finite gap approximation*

The resulting non-dimensional equations (see Appendix A) are for axisymmetric disturbances:

$$(DD_* - a^2)(DD_* - a^2 - i\omega)u_r + \frac{iaQ}{\xi}(DD_* - a^2)b_r - \frac{2a^2V}{r}u_\theta = 0, \qquad (2.1)$$

$$(DD_* - a^2 - i\omega\xi)b_r + ia\xi u_r = 0, \tag{2.2}$$

$$(DD_* - a^2 - i\omega\xi)b_\theta + ia\xi u_\theta - \xi\left(DV - \frac{V}{r}\right)b_r = 0, \tag{2.3}$$

$$(DD_* - a^2 - i\omega)u_\theta + \frac{iaQ}{\xi}b_\theta - (D_*V)u_r = 0. \tag{2.4}$$

In the above system, $D = d/dr$, $D_* = D + 1/r$, $V(r) = Ar + B/r$, with

$$A = \frac{(\mu - \eta^2)}{1 - \eta^2}\frac{\Omega_1 R_2^2}{\nu}, \quad B = \frac{(1 - \mu)}{1 - \eta^2}\frac{\Omega_1 R_1^2}{\nu},$$

and

$$\mu = \Omega_2/\Omega_1 \quad \text{and} \quad \eta = R_1/R_2.$$

Also, $\xi = \nu\sigma\mu_0$ is the magnetic Prandtl number, μ_0 the magnetic permeability, $Q = (B_0 R_2)^2\sigma/\rho\nu$ is the square of the Hartmann number, called the *Chandrasekhar number*, based on the electrical conductivity σ, density ρ, and kinematic viscosity ν of the fluid. The derivation of (2.1)–(2.4) from the basic MHD equations is carried out in Appendix A.

2.1.1. *Boundary conditions*

For the velocity field, no-slip boundary conditions are assumed on the walls. Thus, with continuity (A.7), we have

$$u_r = u_\theta = Du_r = 0, \quad \text{on } r = \eta, 1.$$

The magnetic boundary conditions depend on the electrical properties of the walls. For general finite conducting walls with 3-dimensional disturbances, the relevant boundary conditions were derived by Roberts [16]. As we have specialized in the case of axisymmetric disturbances, the following boundary conditions are obtained. In cylindrical coordinates, the components of the perturbed magnetic field are $(b_r(r), b_\theta(r), b_z(r))$; we then find at $r = 1$ the conditions

$$b_r(1) + ib_z(1)\frac{K_0'(a)}{K_0(a)} = 0,$$

$$b_\theta(1) = \frac{\sigma'}{\sigma}D_*b_\theta(1)\frac{K_0'(a)}{aK_0(a)},$$

where σ' is the conductivity of the walls, and $K_0(ar)$ is the modified Bessel function of the second kind. These conditions come from explicitly solving for the magnetic field in the fluid and in the material medium of the walls and applying the relevant jump conditions on the fields. In the fluid, applying the continuity equation for axisymmetric disturbances,

$$D_*b_r = -iab_z,$$

the boundary conditions stated above become

$$b_r(1) - \frac{K_0'(a)}{aK_0(a)} D_* b_r(1) = 0,$$
$$b_\theta(1) - \frac{\sigma'}{\sigma} \frac{K_0'(a)}{aK_0(a)} D_* b_\theta(1) = 0.$$

Likewise, at $r = \eta$, the relevant conditions are

$$b_r(\eta) - \frac{I_0'(a\eta)}{aI_0(a\eta)} D_* b_r(\eta) = 0,$$
$$b_\theta(\eta) - \frac{\sigma'}{\sigma} \frac{I_0'(a\eta)}{aI_0(a\eta)} D_* b_\theta(\eta) = 0,$$

where $I_0(ar)$ is the modified Bessel function of the first kind. One observes that in those special cases where $\sigma' = 0$, $b_\theta(1) = 0$ and $b_\theta(\eta) = 0$, while for $\sigma' \to \infty$, $D_* b_\theta(1) = 0$ and $D_* b_\theta(\eta) = 0$. For all wall conductivities, the boundary conditions on b_r remain the same.

2.1.2. *Transformed equations*

Equations (2.1)–(2.4) can be transformed by introducing new dependent variables representing the radial and azimuthal components of the magnetic induction vector; these components are denoted by $(\phi(r), \psi(r)) = \frac{i}{a\xi}(b_r(r), b_\theta(r))$ (cf. [2, p. 399]),

$$(DD_* - a^2)(DD_* - a^2 - i\omega)u_r + a^2 Q(DD_* - a^2)\phi + a^2 T\Omega(r)u_\theta = 0, \quad (2.5)$$
$$(DD_* - a^2 - i\omega\xi)\phi - u_r = 0, \quad (2.6)$$
$$(DD_* - a^2 - i\omega\xi)\psi - u_\theta - \frac{\xi}{\kappa r^2}\phi = 0, \quad (2.7)$$
$$(DD_* - a^2 - i\omega)u_\theta + a^2 Q\psi - u_r = 0. \quad (2.8)$$

Here, $\Omega(r) = \left(\frac{1}{r^2} - \kappa\right)$, $\kappa = -\frac{A}{B} = \frac{(1-\mu/\eta^2)}{1-\mu}$, $T = -4AB$ is the *Taylor number.* Thus, when the cylinders rotate in the same direction, $0 < \kappa < 1$. If the outer cylinder is fixed [16], $\kappa = 1$. The boundary conditions are as before with $(\phi(r), \psi(r))$ replacing $(b_r(r), b_\theta(r))$. The limits $Q \to 0$, $\xi \to 0$, give the disturbance equations for Couette flow.

2.2. ***Narrow gap approximation***

In the small gap case $d = R_2 - R_1$ is small compared to R_1, so that terms which are $O(d/R_1)$ can be neglected. A Cartesian coordinate can be introduced. Let

$$x = \frac{r - R_1}{d}.$$

The governing equations become [3]:

$$(D^2 - a^2 - i\omega)(D^2 - a^2)u + a^2 Q(D^2 - a^2)\phi + a^2 T\Omega(x)v = 0, \tag{2.9}$$

$$(D^2 - a^2 - i\omega)v - u + a^2 Q\psi = 0, \tag{2.10}$$

$$(D^2 - a^2 - i\omega\xi)\phi - u = 0, \tag{2.11}$$

$$(D^2 - a^2 - i\omega\xi)\psi - v - \xi\phi = 0, \tag{2.12}$$

where $D = d/dx, T = 2(1-\mu)(\Omega_1 d/\nu)^2 R_1 d, \Omega(x) = 1-(1-\mu)x$, and $Q = (B_0 d)^2\sigma/\rho\nu$. The appropriate boundary conditions are those given by [16]; that is,

$$u = Du = v = \phi - \beta_x D\phi = \psi - \frac{\sigma^{'}}{\sigma}\beta_x D\psi = 0, \quad \text{at } x = 0, 1,$$

for real constants β_0, β_1.

3. Abstract Formulation

For both the finite gap and the narrow gap approximations, one thing is assumed: the magnetic Prandtl number is taken to be zero, that is, $\xi = 0$ in Eqs. (2.5)–(2.8) and (2.9)–(2.12). The first component of the velocity perturbation may then be written in terms of the reduced Laplacian of the magnetic field perturbation. The order of each system is thereby reduced to eight.

3.1. *Finite gap approximation*

The resulting equations, which are based on (2.5)–(2.8), may be written in the finite gap case, with $\xi = 0$ and $s = i\omega$, as

$$(M^*M + sM + a^2Q)u + a^2 T\Omega(r)v = 0, \tag{3.1}$$

$$u + (M_0 + s)v - a^2 Q\psi = 0, \tag{3.2}$$

$$M_{\sigma'}\psi + v = 0, \tag{3.3}$$

where $u = u_r$ and $v = u_\theta$ from (2.5)–(2.8). The operators are defined as follows

$$Mu = (-DD_* + a^2)u := mu, \quad u \in \text{dom } M$$

$$\begin{aligned} M^*Mu &= m^2u, \quad u \in \text{dom } (M^*M) \\ M_{\sigma'}\psi &= m\psi, \quad \psi \in \text{dom } M_{\sigma'} \\ M_0 v &= mv, \quad v \in \text{dom } M_0. \end{aligned}$$

The domains of these operators are contained in a weighted space $\mathfrak{H}_r$, where

$$\mathfrak{H}_r = \left\{\varphi \;\middle|\; \int_\eta^1 r|\varphi|^2 dr < \infty\right\},$$

with scalar product

$$\langle\varphi, \psi\rangle = \int_\eta^1 r\varphi(r)\bar{\psi}(r)dr, \quad \varphi, \psi \in \mathfrak{H}_r$$

and norm

$$\|\varphi\| = \langle \varphi, \varphi \rangle^{1/2}.$$

The domains are given as follows:

$$\begin{aligned}
\text{dom } M^* &= \{\varphi \in \mathfrak{H}_r \mid m\varphi \in \mathfrak{H}_r\}, \\
\text{dom } M &= \big\{\varphi \in \text{dom } M^* \mid \varphi(\eta) = \varphi(1) = D\varphi(\eta) = D\varphi(1) = 0\big\}, \\
\text{dom } M_{\sigma'} &= \Big\{\varphi \in \text{dom } M^* \mid \varphi(\eta) - \frac{\sigma'}{\sigma}\frac{I_0'(a\eta)}{aI_0(a\eta)} D_*\varphi(\eta) \\
&\qquad\qquad = \varphi(1) - \frac{\sigma'}{\sigma}\frac{K_0'(a)}{aK_0(a)} D_*\varphi(1) = 0\Big\}, \\
\text{dom } M_0 &= \big\{\varphi \in \text{dom } M^* \mid \varphi(\eta) = \varphi(1) = 0\big\}.
\end{aligned}$$

The operator M_0 corresponds to $\sigma' = 0$, that is, to perfectly insulating boundary conditions.

With the above definitions in hand, it is not difficult to verify the following properties of the operators just defined [11]. Though the remarks below may not seem completely obvious, they are immediate consequences of standard results in linear differential operator theory (e.g. [14]).

Remark 3.1. M is closed, symmetric, but not maximal and hence not invertible. Moreover, M is positive definite, that is,

$$\langle M\varphi, \varphi \rangle \geq a^2 \|\varphi\|^2, \quad \varphi \in \text{dom } M, \;\; a \neq 0.$$

The case $a = 0$ is excluded because stability is known to hold in this case.

Remark 3.2. M^* is the adjoint of M and has no boundary conditions. The two-dimensional null space of $M^*, \ker M^*$, has the basis

$$\mathbf{q}(r) = \begin{pmatrix} I_1(ar) \\ K_1(ar) \end{pmatrix},$$

where I_1 and K_1 are the modified Bessel functions of order one of the first and the second kind, respectively.

Remark 3.3. $M_{\sigma'}$ is a family of self-adjoint extensions of M. Furthermore, M_0 is a positive definite extension of M. Its resolvent $(M_0 + s)^{-1} := \Gamma(s)$, exists for

$$s \notin \Sigma_a = \big\{s \in \mathbf{C} \mid \text{Re}(s) \leq -a^2, \text{Im}(s) = 0\big\},$$

and $\|\Gamma(s)\|^{-1} > |s + a^2|$, for $\text{Re}(s) > -a^2$ [14, p. 272].

3.2. *Narrow gap approximation*

The function space used now is different from that in the previous section. However, a slight abuse of notion is made in that the same operator M is again used so that a unified exposition results. Here, the corresponding operator formulation is

$$(M^*M + sM + a^2Q)u + a^2T\Omega(x)v = 0, \tag{3.4}$$

$$u + (M_0 + s)v - a^2Q\psi = 0, \tag{3.5}$$

$$M_{\sigma'}\psi + v = 0, \tag{3.6}$$

where

$$\begin{aligned} Mu &= (-D^2 + a^2)u := mu, \quad u \in \text{dom } M, \\ M^*Mu &= m^2u, \quad u \in \text{dom } (M^*M), \\ M_{\sigma'}\psi &= m\psi, \quad \psi \in \text{dom } M_{\sigma'} \\ M_0 v &= mv, \quad v \in \text{dom } M_0. \end{aligned}$$

The domains are contained in $\mathfrak{H}$, where

$$\mathfrak{H} = L^2(0,1) = \left\{ \phi \,\middle|\, \int_0^1 |\phi|^2 dx \right\} < \infty,$$

with scalar product

$$\langle \phi, \psi \rangle = \int_0^1 \phi(x)\bar{\psi}(x)dx, \quad \phi, \psi \in \mathfrak{H},$$

and norm

$$\|\phi\| = \langle \phi, \phi \rangle^{1/2}.$$

The domains are given as follows, with $\psi' = D\psi$:

$$\begin{aligned} \text{dom } M^* &= \{\phi \in \mathfrak{H} \mid m\phi \in \mathfrak{H}\}, \\ \text{dom } M &= \{\phi \in \text{dom } M^* \mid \phi(0) = \phi'(0) = \phi(1) = \phi'(1) = 0\}, \\ \text{dom } M_{\sigma'} &= \left\{ \psi \in \text{dom } M^* | \psi(0) - \frac{\sigma'}{\sigma}\beta_0\psi'(0) = \psi(1) - \frac{\sigma'}{\sigma}\beta_1\psi'(1) = 0 \right\}, \\ \text{dom } M_0 &= \{v \in \text{dom } M^* \mid v(0) = v(1) = 0\}. \end{aligned}$$

The following estimates are not the sharpest possible, but they are sufficient for our purposes. Again, the content of these remarks may not seem completely obvious. Citations are provided where the theory of these differential operators is developed.

Remark 3.4. M is closed, symmetric, but not maximal and hence not invertible. Moreover, M is positive definite, that is, $\langle M\phi, \phi \rangle \geq a^2\|\phi\|^2, \phi \in \text{dom}\, M$, $a \neq 0$ ([14, p. 274]). The case $a = 0$ is excluded because stability is known to hold in that case.

Remark 3.5. M^* is the adjoint of M and has no boundary conditions [14, p. 276]. The two-dimensional null space of $M^*, \ker M^*$, has the basis

$$\mathbf{q}(x) = \begin{pmatrix} e^{ax} \\ e^{-ax} \end{pmatrix}.$$

Remark 3.6. $M_{\sigma'}$ is a family of self-adjoint extensions of M. Furthermore, M_0 is a positive definite extension of M. Its resolvent $(M_0 + s)^{-1} := \Gamma(s)$ exists for $s \notin \Sigma_a = \{s \in \mathbf{C} \mid \mathrm{Re}(s) \leq -a^2, \mathrm{Im}(s) = 0\}$, and $\|\Gamma(s)\|^{-1} > |s + a^2|$, for $\mathrm{Re}(s) > -a^2$ [14, p. 272].

Explicitly, $\Gamma(s)$ is the integral operator such that for $f \in \mathfrak{H}$,

$$\Gamma(s)f = (M_0 + s)^{-1}f = \int_0^1 g(x, z; s)f(z)dz,$$

where

$$g(x, z; s) = \frac{\cosh[\gamma(1 - |x - z|)] - \cosh[\gamma(-1 + x + z)]}{2\gamma \sinh \gamma} \tag{3.7}$$

is the appropriate Green's function and

$$\gamma = \sqrt{a^2 + s}$$

is the branch of the square root which is positive for positive real numbers.

3.3. *General format*

The two formulations, (3.1)–(3.3) and (3.4)–(3.6) are in complete analogy. The usual approach [4] has been to reduce the system from three equations to two by eliminating v. Then the pair of equations becomes, in the current notation,

$$(M^*M + sM + a^2Q)u = a^2T\Omega M_{\sigma'}\psi, \tag{3.8}$$

$$(M_0M_{\sigma'} + sM_{\sigma'} + a^2Q)\psi = u. \tag{3.9}$$

When numerical techniques are employed, these are the natural equations to be considered. In particular, these equations contain, in the insulating case where $\sigma' = 0$, Dirichlet boundary conditions on ψ, and in the perfect conducting case, where $\sigma' \to \infty$, Neumann boundary conditions on ψ. In all cases, since $M_{\sigma'}\psi = -v$, $M_{\sigma'}\psi$ satisfies Dirichlet boundary conditions. Significantly, in (3.9) we see the presence of the factored operator $(M_0 + s)M_{\sigma'}$ operating on ψ; these factors commute if $\sigma' = 0$, which permits a simplification in the theory.

In the special case where $\sigma' = 0$, the variable ψ will be eliminated from (3.1)–(3.3) and (3.4)–(3.6). Since, in this case, $\psi = -M_0^{-1}v$ (Remarks 3.3 and 3.6), the result is

$$(M^*M + sM + a^2Q)u = -a^2T\Omega v, \tag{3.10}$$

$$(M_0 + s + a^2QM_0^{-1})v = -u. \tag{3.11}$$

One notices that the formulation (3.10)–(3.11) may also be obtained directly from (2.1)–(2.4) by first setting $\sigma' = 0$, eliminating b_r and b_θ, and then taking the limit as $\xi \to 0$. When $Q = 0$, this yields immediately the classical stability equations for Couette flow. Significantly, this formulation also lends itself to the verification of the PES [11].

The idea of the method of solution is based on the notion of a *positive* operator [9, 13], which is the generalization of a positive matrix, that is, one with all of its entries positive. Such matrices have the property that they possess a greatest positive eigenvalue which is identical to the spectral radius. To apply the method, the resolvent of the linearized stability operator is analyzed. This resolvent is in the form of compositions of certain integral operators. When the Green's function kernels for these operators are all nonnegative, the resulting operator is termed positive. The infinite dimensional counterpart of this property is contained in the following theorem.

Theorem 3.1. **[15]** *If a linear, compact operator A, leaving invariant a cone $\mathfrak{C}$, has a point of the spectrum different from zero, then it has a positive eigenvalue λ, not less in modulus than every other eigenvalue, and to this number corresponds at least one eigenvector $\phi \in \mathfrak{C}$ of the operator A ($A\phi = \lambda\phi$), and at least one eigenvector $\psi \in \mathfrak{C}^*$ of the operator A^*.*

For these problems the cone consists of the set of nonnegative functions.

It is now possible to write our system as a single equation in u. Eliminating v from (3.11), it follows that

$$v = -(M_0 + s + a^2 Q M_0^{-1})^{-1} u = -F(s)u, \tag{3.12}$$

where

$$F(s) = (M_0 + s + a^2 Q M_0^{-1})^{-1}. \tag{3.13}$$

Similarly, in (3.10),

$$u = -(M^* M + sM + a^2 Q)^{-1} a^2 T \Omega v = -G(s) a^2 T \Omega v, \tag{3.14}$$

where

$$G(s) = (M^* M + sM + a^2 Q)^{-1}. \tag{3.15}$$

So, substituting for v, from (3.12) in (3.14), we obtain

$$u = a^2 T G(s) \Omega F(s) u.$$

In a more compact form this equation may be written as

$$[I - K(s)]u = 0, \tag{3.16}$$

where

$$K = a^2 T G(s) \Omega F(s).$$

This formal derivation of (3.16) will be justified shortly.

3.4. *The principle of exchange of stabilities*

The operator to be studied in what follows is $[I - K(s)]^{-1}$. Suppose $K(s)$ depends analytically on s in a certain right half of the complex plane, for instance $\mathrm{Re}(s) > \alpha$. Furthermore, for all s_0 for which $K(s)$ is defined,

$$[I - K(s)]^{-1} = \left\{I - [I - K(s_0)]^{-1}[K(s) - K(s_0)]\right\}^{-1}[I - K(s_0)]^{-1}. \qquad (3.17)$$

So, if for all real s_0 greater than some fixed constant α

(P1) $[I - K(s_0)]^{-1}$ is positive,

and

(P2) $K(s)$ has a power series about s_0 in $(s_0 - s)$ with positive coefficients, i.e. $(-d/ds)^n K(s_0)$ is positive for all n, then the right side of (3.17) has an expansion in $(s_0 - s)$ with positive coefficients.

Some of the implications of these hypotheses are developed now. Notice that by (**P1**), $[I - K(s)]$ is invertible for $\mathrm{Re}(s)$ sufficiently large. Consider the related eigenvalue problem

$$K(s)\phi = \lambda(s)\phi.$$

With (**P2**), when s is real, $K(s)$ is a nonnegative integral operator, so that by the Krein–Rutman theorem, it has a positive eigenvalue λ, which is an upper bound for the absolute values of all the eigenvalues, and the corresponding eigenfunction $\phi(s)$ is nonnegative. We observe that if $\lambda(s) > 1$, then

$$[I - K(s)][-\phi(s)] = (\lambda - 1)\phi \geq 0,$$

while $-\phi \leq 0$. Thus, if $[I - K(s)]$ is nonnegative, then $\lambda < 1$. Conversely, if $\lambda(s) < 1$, then the spectral radius of K is less than 1, so that

$$[I - K(s)]^{-1} = I + K + K^2 + \cdots,$$

which is nonnegative. Thus, it is that $\lambda(s)$ is continuous and non-increasing. When there is a value of s, with $\mathrm{Re}(s) > 0$, where $[I - K(s)]$ is not invertible, then $\lambda(s) \geq 1$, and hence there must be a real value s_1 such that $\lambda(s_1) = 1$ and $\lambda(s) < 1$, when s is real and $s > s_1$. So the methods of [19] apply, showing that "there exists a real eigenvalue

$$s_1 \leq \alpha \qquad (3.18)$$

such that the spectrum of $K(s)$ lies in the set $\{s \mid \mathrm{Re}(s) \leq s_1\}$". This is equivalent to the PES.

To verify conditions (**P1**) and (**P2**), the structure of the operators $F(s)$ and $G(s)$ will be examined. Previously, the following results were obtained, without the magnetic field. The finite gap case (cylindrical coordinates) was treated in [11] and the Cartesian case, directly applicable to the narrow gap case was treated in [12];

there the following were proved:

Lemma 3.1. *The operator* $\Gamma(s) = (M_0 + s)^{-1}$ *is a positive operator for all real* $s > -a^2$, *and* $\Gamma(s)$ *has a power series expansion about* s_0 *in* $(s_0 - s)$ *with positive coefficients, i.e.* $\left(-\frac{d}{ds}\right)^n \Gamma(s_0)$ *is positive for* s_0 *real, for all* n.

Lemma 3.2. *The operator* $L(s) = (M^*M + sM)$ *has a positive inverse for all real* $s > -a^2$; *this inverse has a power series expansion about* s_0 *in powers of* $s_0 - s$ *with positive expansion coefficients.*

Thus, with the Lemmas 3.1 and 3.2 and Remarks 3.3 and 3.6, an analysis of (3.16) is possible. The remaining details require a consideration of (3.13) and (3.15). One notices that these operators are perturbations of their counterparts for $Q = 0$. However, as the calculations of [4] have shown, when Q becomes large, simple perturbation arguments may fail. Thus, the structure of $F(s) = (M_0 + s + a^2QM_0^{-1})^{-1}$ must first be examined. Note that $F(0)$ is a positive operator as an analytic continuation of $(M_0 + \tau(a^2QM_0^{-1}))^{-1}$ from $\tau = 0$ to $\tau = 1$. Since $F(0) = (M_0 + a^2QM_0^{-1})$ is self-adjoint and positive definite, we also have $\langle (M_0 + a^2QM_0^{-1})\varphi, \varphi\rangle \geq a^2\|\varphi\|^2, \varphi \in \text{dom } M_0, a \neq 0$. Thus, by Remarks 3.3 and 3.6, $F(s)$ will exist, for all Q, on the same region in the s-plane as $(M_0+s)^{-1} = \Gamma(s)$, that is, for $s \notin \Sigma_a = \{s \in \mathbf{C} \mid \text{Re}(s) \leq -a^2, \text{Im}(s) = 0\}$. Thus, we may write

$$\begin{aligned} F(s) &= (M_0 + s_2 + a^2QM_0^{-1})^{-1}\big(I + (s - s_2)(M_0 + s_2 + a^2QM_0^{-1})^{-1}\big)^{-1} \\ &= F(s_2)\big(I + (s - s_2)F(s_2)\big)^{-1}, \end{aligned} \tag{3.19}$$

where s_2 is real and fixed, $s_2 > -a^2$. This representation is legitimate since $F(s_2)$ exists and is a positive operator as an analytic continuation of $F(s)$ from $s = 0$ to $s = s_2$. Now, the type of expansion needed to analytically continue $F(s)$ as a positive operator is developable precisely because M_0^{-1} and $F(s)$ commute. The result is that

$$\frac{d}{ds}F(s) = -(F(s))^2, \quad \text{etc.}$$

Likewise, for $G(s) = (M^*M + sM + a^2Q)^{-1}$, by arguments used to prove Lemma 3.2 [11, 12], it can be shown that $G(s)$ exists on the same region in the s-plane as $(M^*M + sM)^{-1} = L^{-1}(s)$, and $(M^*M + a^2Q)^{-1} = G(0)$ is a positive operator as an analytic continuation of $(M^*M + \tau(a^2Q))^{-1}$ from $\tau = 0$ to $\tau = 1$. Hence, it is possible to write

$$\begin{aligned} G(s) &= (M^*M + s_3M + a^2Q)^{-1}\big(I + (s - s_3)M(M^*M + s_3M + a^2Q)^{-1}\big)^{-1} \\ &= G(s_3)\big(I + (s - s_3)MG(s_3)\big), \end{aligned} \tag{3.20}$$

where s_3 is real and fixed, $s_3 > -a^2$. This representation is legitimate since $G(s_3)$ exits and is a positive operator as an analytic continuation of $G(s)$ from $s = 0$ to $s = s_3$, by the methods of [11] or [12] that are involved in the proof of Lemma 3.2. These same methods may be again employed to show that $-\frac{d}{ds}G(s_3)$ is a positive operator, etc. Based on the observations (3.19) and (3.20), since the choices of s_2 and s_3 in order to prove the results are arbitrary and the regions of existence of $F(s)$ and $G(s)$ have been established, the region of positivity may be extended by analytic continuation to all real $s_0 > -a^2$. The preceding arguments are summarized as the following lemma.

Lemma 3.3. *The operators* $F(s)$, (3.13), *and* $G(s)$, (3.15), *are positive operators for all real* $s > -a^2$. $F(s)$ *and* $G(s)$ *have power series expansions about* s_0 *in* $(s_0 - s)$ *with positive coefficients, i.e.* $\left(-\frac{d}{ds}\right)^n F(s_0)$ *and* $\left(-\frac{d}{ds}\right)^n G(s_0)$ *are positive for* s_0 *real,* $s_0 > -a^2$, *for all* n.

The purpose of Lemma 3.3 was to validate the hypotheses (**P1**) and (**P2**), the latter involving expansions of $F(s)$ and $G(s)$ so that $K(s)$ may be represented by power series. Hypothesis (**P1**) involves $\|K(s)\|$, which depends on all of the parameters in the equations, yet it can be managed, as the discussion of (3.18) has indicated. Incipient instability will occur when T and Q are sufficiently large. What is emerging is that the first eigenvalue will be real as the exchange of stabilities occurs. The desired result is now apparent and the conditions under which PES are to be expected can be stated as follows:

Theorem 3.2. *The Principle of Exchange of Stabilities holds for* (3.10)–(3.11) *when the cylinders rotate in the same direction, the circulation decreases outwards, and the cylinders have insulating walls. This result holds for both the finite gap and the narrow gap approximations.*

Proof. The system (3.10)–(3.11) may be written as the single equation suggested by (3.16),

$$u = K(s)u, \tag{3.21}$$

where

$$K(s) = a^2 T G(s) \Omega F(s).$$

In the finite-gap case, $\Omega = \Omega(r) = \left(\frac{1}{r^2} - \kappa\right), \eta \le r \le 1$. In the narrow gap case, $\Omega = \Omega(x) = 1 - (1-\mu)x, 0 \le x \le 1$.

The resolvent as defined in (3.17) must now be examined. It has been demonstrated that the original system (3.10)–(3.11), and the transformed system (3.21), have spectra that agree except on a set which is a subset of the real half-line. We have shown in Lemma 3.3 that $F(s)$ and $G(s)$ have power series expansions for real $s_0 > -a^2$. To apply the Krein–Rutman Theorem, we can take the Banach space to be the Hilbert space $\mathfrak{H}$ or $\mathfrak{H}_r$; $\mathfrak{C}$, the cone of nonnegative functions in $\mathfrak{H}$ or $\mathfrak{H}_r$; and the positive operator A composed of the operators $\left(-\frac{d}{ds}\right)^n F(s_0), \left(-\frac{d}{ds}\right)^n G(s_0), n = 0, 1, \ldots,$ which are compact and linear.

To verify condition (**P2**), we again note that Ω does not change sign, while a^2 and T are positive. Therefore, by the product rule for differentiation, one concludes that $K(s)$ in (3.21) satisfies condition (**P2**).

It has been demonstrated that all of the terms in $K(s)$ determine positive operators. Moreover, for s real and sufficiently large, by Remarks 3.3 and 3.6, and expressions (3.19) and (3.20), the norms of the operators $F(s)$ and $G(s)$ become arbitrarily small. Hence, $\|K(s)\|$ will be less than 1 for real s sufficiently large. Then $[I - K(s)]^{-1}$ has a convergent Neumann series and, hence, is positive. This is the content of condition (**P1**). □

4. Concluding Discussion

The main purpose of this paper has been to ascertain under what conditions the principle of exchange of stabilities (PES) might hold for hydromagnetic Couette flow. There has been renewed interest in this problem because of the possibility of experimentally demonstrating magnetorotational instability (MRI), which has been predicted theoretically [10]. The regime studied most successfully in this article was that for which the wall boundary conditions are insulating, and it is likely that in a laboratory experiment, the best set-up would have insulating walls on the cylinders. The conclusion to be drawn here and also from the results of previous workers [4, 7, 18] is that the PES may not always hold in the conducing-wall case. Truely, in MHD, researchers would like to know under what conditions marginal stability means $s = 0$ (PES), rather than $\mathrm{Re}(s) = 0$. These results show that only in the insulating-wall case can one be sure that this is true for all Chandrasekhar numbers Q, not just $Q = 0$ (non-magnetic). In fact, the computations just cited show that with any conductivity, for Q sufficiently large, there will be no PES. The analysis just completed here shows why this is to be expected from operator theory; there is a lack of commutativity of the underlying operators. In future work, we hope to extend the analysis to nonzero values of magnetic Prandtl number ξ.

Appendix A. Derivation of the Linearized Disturbance Equations

For completeness, Eqs. (2.1)–(2.4) are derived here, although much the same derivation can be found in [2, 16] or [10]. The equations of incompressible MHD are

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{v} = \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B},$$
$$\nabla \cdot \mathbf{B} = 0,$$
$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{1}{\rho} \nabla P - \frac{1}{\mu_0 \rho} \mathbf{B} \cdot \nabla \mathbf{B} = \nu \nabla^2 \mathbf{v},$$
$$\nabla \cdot \mathbf{v} = 0,$$

where $P := \bar{P} + \mathbf{B} \cdot \mathbf{B}/2\mu_0$ is the hydrodynamic pressure plus the magnetic pressure. The basic state is circumferential motion, $\mathbf{v}_0 = r\Omega(r)\mathbf{e}_\theta$, with a constant axial magnetic field, $\mathbf{B} = B_0 \mathbf{k}$. Linearized axisymmetric perturbations to the velocity, magnetic field and pressure, respectively, $\mathbf{u}$, $\mathbf{b}$ and p satisfy

$$\frac{\partial b_r}{\partial t} - B_0\frac{\partial u_r}{\partial z} = \frac{1}{\mu_0\sigma}\left(\nabla_a^2 - \frac{1}{r^2}\right)b_r, \tag{A.1}$$

$$\frac{\partial b_\theta}{\partial t} - B_0\frac{\partial u_\theta}{\partial z} - r\frac{d\Omega}{dr}b_r = \frac{1}{\mu_0\sigma}\left(\nabla_a^2 - \frac{1}{r^2}\right)b_\theta, \tag{A.2}$$

$$\frac{\partial u_r}{\partial t} - 2\Omega u_\theta + \frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{B_0}{\mu_0\rho}\frac{\partial b_r}{\partial z} = \nu\left(\nabla_a^2 - \frac{1}{r^2}\right)u_r, \tag{A.3}$$

$$\frac{\partial u_\theta}{\partial t} + D_*(r\Omega)u_r - \frac{B_0}{\mu_0\rho}\frac{\partial B_\theta}{\partial z} = \nu\left(\nabla_a^2 - \frac{1}{r^2}\right)u_\theta, \tag{A.4}$$

$$\frac{\partial u_z}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{B_0}{\mu_0\rho}\frac{\partial b_z}{\partial z} = \nu\nabla_a^2 u_z, \tag{A.5}$$

$$\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)b_r + \frac{\partial b_z}{\partial z} = 0, \tag{A.6}$$

$$\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)u_r + \frac{\partial u_z}{\partial z} = 0, \tag{A.7}$$

where $\nabla_a^2 = \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$ is the axisymmetric Laplacian, and $D_* = \frac{d}{dr} + \frac{1}{r}$. The parameters ρ, μ_0, σ, and ν are taken to be constants. Applying $\partial/\partial r + 1/r$ to (A.3) and $\partial/\partial z$ to (A.5) and adding the results gives

$$\frac{1}{\rho}\nabla_a^2 p = D_*(2\Omega u_\theta),$$

because of the solenoidal conditions (A.6) and (A.7). Applying $\partial/\partial r$ again, this becomes

$$\left(\nabla_a^2 - \frac{1}{r^2}\right)\Pi = 2\Omega\frac{\partial^2 u_\theta}{\partial z^2}, \tag{A.8}$$

where $\Pi = 2\Omega u_\theta - \frac{1}{\rho}\partial p/\partial r$. Hence, the radial component of the momentum equations (A.3), may be restated as

$$\frac{\partial u_r}{\partial t} - \Pi - \frac{B_0}{\mu_0\rho}\frac{\partial b_r}{\partial z} = \nu\left(\nabla_a^2 - \frac{1}{r^2}\right)u_r. \tag{A.9}$$

Dimensionless variables are introduced as

$$\tilde{t} = \frac{t}{(R_2^2/\nu)}, \quad \tilde{r} = \frac{r}{R_2}, \quad \tilde{\mathbf{u}} = \frac{\mathbf{u}}{(\nu/R_2)}, \quad \tilde{\mathbf{b}} = \frac{\mathbf{b}}{B_0}, \quad \text{etc;}$$

the *Chandraskehar number* $Q = (B_0R_2)^2\sigma/\rho\nu$, and $\xi = \nu\sigma\mu_0$ is the *magnetic Prandtl number.*

The velocity field is decomposed as $\tilde{\mathbf{v}} = \hat{\mathbf{v}}(r)e^{i(\tilde{\omega}\tilde{t}+\tilde{a}\tilde{z})}$, and similarly for the magnetic field $\tilde{\mathbf{b}} = \hat{\mathbf{b}}e^{i(\tilde{\omega}\tilde{t}+\tilde{a}\tilde{z})}$, and $\tilde{\Pi} = \hat{\Pi}e^{i(\tilde{\omega}\tilde{t}+\tilde{a}\tilde{z})}$. Next, $\hat{\Pi}$ may be eliminated by substituting (A.9) into (A.8). The tildes and carets are dropped on all quantities and the resulting equation is (2.1). Likewise, (A.1) reduces to (2.2), (A.2) reduces to (2.3), and (A.4) reduces to (2.4).

References

[1] S. Chandrasekhar, The stability of viscous flow between rotating cylinders in the presence of a magnetic field. *Proc. Roy. Soc. Edinburgh Sec. A* **216** (1953), 293–309.
[2] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability.* Oxford University Press, London, 1961.
[3] T. S. Chang and W. K. Sartory, On the onset of instability by oscillatory modes in hydromagnetic Couette flow, *J. Fluid Mech.* **301** (1967), 451–471.
[4] C.-K. Chen and M. H. Chang, Stability of hydromagnetic dissipative Couette flow with non-axisymmetric disturbance, *J. Fluid Mech.* **366** (1998), 135–158.
[5] P. A. Davidson, *An Introduction to Magnetohydrodynamics*, Cambridge University Press, Cambridge, 2001.
[6] S. H. Davis, On the principle of exchange of stabilities, *Proc. Roy. Soc. Edinburgh Sec. A* **310** (1969), 341–358.
[7] G. P. Galdi, Nonlinear stability of hydromagnetic Couette flow, in *Computers and Experiments in Fluid Flow*, eds. G. M. Carlomagno and C. A. Brebbia, Springer-Verlag, Berlin, 1989, pp. 183–199.
[8] G. P. Galdi and B. Straughan, Exchange of stabilities, symmetry and nonlinear stability, *Arch. Rat. Mech. Anal.* **89** (1985), 211–228.
[9] F. R. Gantmacher, *Applications of the Theory of Matrices*, Interscience, New York, 1959.
[10] J. Goodman and H. Ji, Magnetorotational instability of dissipative Couette flow, *J. Fluid Mech.* **262** (2002), 365–382.
[11] I. H. Herron and H. N. Ali, The principle of exchange of stabilities for Couette flow, *Quart. Appl. Math.* **61** (2003), 279–293.
[12] I. H. Herron, On the principle of exchange of stabilities in Rayleigh–Bénard convection, II — No-slip boundary conditions, *Ann. Univ. Ferrara Sez. VII (N.S.)* **49** (2003), 169–182.
[13] S. Karlin, Positive operators, *J. Math. Mech.* **8** (1959), 907–937.
[14] T. Kato, *Perturbation Theory for Linear Operators*, 2nd ed. Springer-Verlag, Berlin, 1976.
[15] M. G. Krein and M. A. Rutman, Linear operators leaving invariant a cone in a Banach space, *Uspehi Matem. Nauk* **3** (1948), 3–95 (A.M.S. Transl. No. 26 (1950)).
[16] P. H. Roberts, The stability of hydromagnetic Couette flow, *Proc. Camb. Phil. Soc.* **60** (1964), 635–651.
[17] P. H. Roberts, *An Introduction to Magnetohydrodynamics.* American Elsevier, New York, 1967.
[18] P. Tabeling, Magnetohydrodynamic Taylor vortex flows, *J. Fluid Mech.* **112** (1981), 329–345.
[19] H. F. Weinberger, Exchange of stability in Couette flow, in *Bifurcation Theory and Nonlinear Eigenvalue Problems*, eds. J. B. Keller and S. Antman, Benjamin, New York, 1969.