Microlocal Analysis in Radar Imaging

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Outline

• How microlocal analysis is used in radar imaging
• Examples:
  ▸ Standard wide-angle SAR
  ▸ Doppler SAR (background for last talk!)
  ▸ SAR with multiple scattering
Usual Procedure

- Develop (linearized) mathematical model for radar data in the form of a Fourier Integral Operator (FIO) or sum of FIOs
- Form image as filtered adjoint (also FIO) applied to data
- Analyze relation between image and scene
  - want image fidelity operator to be pseudodifferential operator
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Synthetic-Aperture Radar (SAR)
Reconstruct a function from its integrals over circles
Antenna moves on path $\gamma(s)$

Write $R_{s,x} = |\gamma(s) - x|$ data is of the form

$$d(t, s) = \int \int e^{-i\omega(t - 2|R_{s,x}|/c)} A(\omega, s, x) d\omega V(x) dx =: F[V](t, s)$$

$A$ includes factors for:
1. geometrical spreading
2. antenna beam patterns
3. waveform sent to antenna

reflectivity function (scene)
Construction of imaging operator

recall
\[ d(s, t) = \int \int e^{-i\omega(t-2|\mathbf{R}_{s, x}|/c)} A(\omega, s, \mathbf{x}) d\omega V(\mathbf{x}) d\mathbf{x} \]

image = \( Bd \) where
\[ Bd(z) = \int \int e^{i\omega(t-2|\mathbf{R}_{s, z}|/c)} Q(z, s, \omega) d\omega \ d(s, t) ds dt \]

where \( Q \) is to be determined.

- \( B \) has phase of \( F^* \) (\( L^2 \) adjoint)
- Compare:
  - inverse Fourier transform
  - inverse Radon transform
- This approach often results in exact inversion formula

microlocal analysis interpretation: joint work with Cliff Nolan
Analysis of approximate inverse of $F$

$$I(z) = \int e^{i\omega(t-2|R_s,z|/c)} Q(z, s, \omega) d\omega \, d(s, t) d\sigma$$

where $Q$ is to be determined below.

- Plug in expression for the data and do the $t$ integration:

$$I(z) = \int \int e^{i2k(|R_s,z|-|R_s,x|)} QA(\ldots) d\omega ds V(x) d^2x$$

point spread function

- Want $K$ to look like a delta function

$$\delta(z - x) = \int e^{i(z-x)\cdot \xi} d\xi$$

- Analyze $K$ by the method of stationary phase
\[ K(z, x) = \int e^{i2k(|R_{s,z}| - |R_{s,x}|)} QA(\ldots) d\omega ds \]

main contribution comes from

**critical points**

\[ |R_{s,z}| = |R_{s,x}| \]
\[ \hat{R}_{s,z} \cdot \dot{\gamma}(s) = \hat{R}_{s,x} \cdot \dot{\gamma}(s) \]

If \( K \) is to look like
\[ \delta(z - x) = \int e^{i(z-x) \cdot \xi} d^2\xi, \]
we want critical points only when \( z = x \).

Antenna beam

should illuminate only one of the critical points ⇒ use side-looking antenna
\[ K(z, x) = \int e^{i2k(|R_s,z|-|R_s,x|)}QA(\ldots)d\omega ds \]

At critical point \( z = x \):

1. Do Taylor expansion of exponent about \( z = x \):

\[
2k(|R_s,z| - |R_s,x|) = (z - x) \cdot \Xi(x, z, s, \omega)
\]

near \( z = x \), \( \Xi(x, z, s, \omega) \approx 2k[\hat{R}_{s,z}]_T \)

2. Make (Stolt) change of variables

\[
(s, \omega) \rightarrow \xi = \Xi(x, z, s, \omega)
\]

Then

\[
K(z, x) = \int e^{i(z-x) \cdot \xi}QA(\ldots) \left| \frac{\partial(s, \omega)}{\partial \xi} \right| d^2 \xi
\]

Take \( Q = 1/(A |\partial(s, \omega)/\partial\xi|) \).

\[ \uparrow \]

*Beylkin determinant.*
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Doppler SAR 

- transmit HRR waveform
- at time t, get range slice
- translate, get different views
- use tomographic reconstruction

Doppler-only SAR

- transmit HDR waveform
- at frequency f, get Doppler slice
- translate, get different views
- use tomographic reconstruction

joint with B. Borden
Mathematical model

transmit CW waveform (no pulsing, no modulation)

\[ E^{sc}(t, x) \propto \int e^{-i\omega_0(t-2|x-y|/c)} A(\omega_0, x, y) \rho(y) dy \]

antenna position

let \( x \rightarrow \gamma(t) \)

\[ s^{sc}(t) = E^{sc}(t, \gamma(t)) \propto \int e^{i\omega_0[t-2|\gamma(t)-y|/c]} A(\omega_0, t, y) \rho(y) dy \]

received signal

antenna beam

gеometrical spreading

scattering density
receiver: matched filter is windowed Fourier transform

\[ \eta(\tau, \omega) = \int s^{sc}(t) e^{i \omega (t - \tau)} \psi(t - \tau) dt \]

- Window center (slow time)
- Doppler frequency
- Window

\[ \gamma(t) = \gamma(\tau) + \dot{\gamma}(\tau)(t - \tau) + \cdots \]

\[ |\gamma(t) - y| = |\gamma(\tau) - y + \dot{\gamma}(\tau)(t - \tau)| \]

\[ = |\gamma(\tau) - y| + (\gamma(\tau) - y) \cdot \dot{\gamma}(\tau)(t - \tau) + \cdots \]

\[ \hat{R}_{y, \tau} \]
Fourier integral operator

stationary phase $\rightarrow$ leading order contribution is from

$$0 = \frac{d\text{phase}}{dt} = (\omega - \omega_0) + 2\omega_0 \hat{R}_{y,\tau} \cdot \dot{\gamma}(\tau)/c$$
Image formation

data:

\[ \eta(\tau, \omega) \approx \int \int e^{it[(\omega - \omega_0) + 2\omega_0 \hat{R}_y, \tau \cdot \hat{\gamma}(\tau)/c]} A(\tau, t, y) dt \rho(y) dy \]

form image via:

\[ I(x) = \int e^{-it\phi(\omega, \tau, x)} B(t, \omega, \tau, x) dt \eta(\tau, \omega) d\tau d\omega \]
Numerical tests

\[ \omega_0 = 100 \text{MHz} \]

\[ v = 8 \text{m/s} \]

.5s sliding time window
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SAR with Multiple Scattering

single scattering

antenna

double-bounce scattering

joint work with Bob Bonneau
Field from source:

\[ G_0(\omega, x, y') = \frac{e^{ik|x-y'|}}{4\pi|x-y'|} \]

\[ k = \omega/c_0 \]

The multiple-scattering Green's function from a single extra point \( z \):

\[ G_1(\omega, x, y') = G_0(\omega, x, y') + G_0(\omega, x, z)\mu G_0(\omega, z, y') \]

\[ G_1^{sc} \]

\[ e^{ik|x-z|} \quad e^{ik|z-y'|} \]

\[ \mu \frac{e^{ik|x-z|}}{4\pi|x-z|} \quad \frac{e^{ik|z-y'|}}{4\pi|z-y'|} \]
Born-approximated time-domain field is

\[ g_B(t, y, y') = g_1(t, y, y') + \int e^{-i\omega t} G_1(\omega, y, x) q(x) G_1(\omega, x, y') \omega^2 d\omega dx \]

\[ = g_0 + g_1^{\text{sc}} + \int e^{-i\omega t} (G_0 + G_1^{\text{sc}}) q(G_0 + G_1^{\text{sc}}) \omega^2 d\omega dx \]

\[ = g_0 + g_1^{\text{sc}} + (F_1 + F_2 + F_3 + F_4)[q] \]

Fourier transform of \( G_1 \) scatterer to be imaged

\[ F_1[q](t, y, y') = \int e^{-i\omega t} G_0(\omega, y, x) G_0(\omega, x, y') q(x) d\omega dx \]
\[ F_2[q](t, y, y') = \int e^{-i\omega t} G_1^{\text{sc}}(\omega, y, x) G_0(\omega, x, y') q(x) d\omega dx \]
\[ F_3[q](t, y, y') = \int e^{-i\omega t} G_0(\omega, y, x) G_1^{\text{sc}}(\omega, x, y') q(x) d\omega dx \]
\[ F_4[q](t, y, y') = \int e^{-i\omega t} G_1^{\text{sc}}(\omega, y, x) G_1^{\text{sc}}(\omega, x, y') q(x) d\omega dx \]
Travel times

\[ F_1[q](t, y, y') = \int e^{-i\omega t} G_0(\omega, y, x)G_0(\omega, x, y')q(x)d\omega dx \]

\[ G_0(\omega, x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|} \]

\[ \Rightarrow F_1 \text{ is of the form} \]

\[ F_j[q](t, y, y') = \int e^{-i\omega[t-\tau_j(y,y',x)]}a_j(\omega, y, y', x)d\omega q(x)dx \]

\[ \tau_1(y, y', x) = (|y - x| + |x - y'|)/c_0 \]

\[ \tau_2(y, y', x) = (|y - z| + |z - x| + |x - y'|)/c_0 \]

\[ \tau_3(y, y', x) = (|y - x| + |x - z| + |z - y'|)/c_0 \]

\[ \tau_4(y, y', x) = (|y - z| + 2|x - z| + |z - y'|)/c_0 \]

Fourier integral operator!
Imaging Strategy

1. Find scatterer(s) in foreground from early-time data

2. Subtract out $g_1(t, y, y')$ corresponding to (now known) scatterer at position $z$

3. Form image by filtered backprojection along appropriate paths:

$$I(p) = B[g^\text{sc}](p) := \sum_{j \in \{\text{paths}\}} \int e^{i\omega[t-\tau_j(y', y', p)]} b_j(\omega, p, y, y') g^\text{sc}(t, y, y') d\omega dt dy dy'$$
The forward operator is a sum of terms of the form
\[ F_j[q](t, y, y') = \int e^{-i\omega [t-\tau_j(y, y', x)]} a_j(\omega, y, y', x) d\omega q(x) dx \]

and the backprojection operator is its (filtered) adjoint
\[ B[g^{sc}](p) := \sum_{j \in \{\text{paths}\}} \int e^{i\omega [t-\tau_j(y, y', p)]} b_j(\omega, p, y, y') g^{sc}(t, y, y') d\omega dt dy dy' \]

Expect adjoint to provide approximate inverse

Compare:
Fourier transform: inverse = adjoint
Radon transform: inverse = filtered adjoint
Apply backprojection operator to data (i.e., compose backprojection operator with forward map)

\[ I(p) \approx \sum_{i=1}^{4} \sum_{j=1}^{4} B_i[F_j[q]](p) = \int K(p, x)q(x)dx, \]

Want to have

\[ K(p, x) = \delta(p - x) \propto \int e^{i\xi \cdot (p-x)}d\xi \]

The contribution to \( K \) from \( B_i F_j \) is

\[ K_{i,j}(p, x) = \int e^{i\omega(t-\tau_i(y,y',p))}b_i(\omega, p, y, y') \]

\[ \times e^{-i\omega'(t-\tau_j(y,y',x))}a_j(\omega', y, y', x)d\omega d\omega' dt dy dy' \]
Diagonal terms \( B_i F_i, i \leq 3 \)

- backprojected path = path in forward operator
- \( p=x \) is the only critical point
- change of variables \( \Rightarrow \)

\[
B_i F_i(p, x) = \int e^{i \xi \cdot (p-x)} b_i(\xi, p) a_i(\xi, x) J(\xi, p, x) d\xi d\xi
\]

\( \Rightarrow \) choice of filter \( b_i \)

\( B_i F_i \) pseudodifferential operator \( \Rightarrow \) image preserves visible edges
The change of variables

phase of $K_{i,i}$ is

$$\omega[\tau_i(y, y', x) - \tau_i(y, y', p)] = (x - p) \cdot \Xi^i(p, x, y, y', \omega);$$

where

$$\Xi^1(p, p, y, y', \omega) = k[p - y + p - y'];$$
$$\Xi^2(p, p, y, \omega) = k[p - y + p - z];$$
$$\Xi^3(p, p, y', \omega) = k[p - z + p - y'].$$

Resolution is determined by these vectors
(region in Fourier space)
Resolution

$K_{1,1}$, single bounce

$K_{2,2}$

$K_{3,3}$
The term $B_4 F_4$

gives rise to artifacts

Do not include this term in backprojection.
Backprojection should involve only terms with a direct path to object.
Off-diagonal terms

artifacts can appear here
Conclusions

• model can easily include antenna beam pattern and waveform
• complicated bookkeeping not needed
• backprojection operator should include only terms with a direct path to object
• resulting image reproduces visible singularities
• including multipath does improve resolution
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Open Problems

• Multiple scattering
• Data of lower dimensionality than scene
• Scenes with unknown motion