INTRODUCTION TO DIFFERENTIAL EQUATIONS, Test 2A
Sections 13 - 16, Spring 2010

Section ___________ Name _____

Instructions. You are allowed to use one double-sided 8 1/2 x 11 inch sheet of notes. No books or electronic equipment (including calculators, PDAs, computers, cell phones) are allowed. Do not collaborate in any way. In order to receive credit, your answers must be clear and legible. In case of an error in a test question, simply write in the correct answer. Except for # 8, all questions are 6 points each.

Problems 1-3 deal with the two-point boundary value problem

\[ X'' + \lambda X = 0, \quad X(0) = 0, \quad X'(\pi) = 0 \]  \( \text{(8)} \)

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1. When \( \lambda = 0 \), the nonzero solutions of (8) are
A) \( X(x) = Ax + B \)  \quad B) \( X(x) \propto x \)  \quad C) \( X(x) \propto 1 \)  \quad D) \( X(x) = A \cos nx + B \sin nx \)
E) The \( \lambda = 0 \) case has no nonzero solutions.

\[ X'' = 0 \quad \Rightarrow \quad X = c_1 x + c_2 \quad \Rightarrow \quad X(0) = 0 \quad \Rightarrow \quad X'(\pi) = c_2 \]

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2. When \( \lambda < 0 \), the problem (8) has nonzero solutions
A) only when \( \lambda = -n^2, n = 1, 2, \ldots \); in that case \( X(x) = A e^{nx} + B e^{-nx} \)
B) only when \( \lambda = -n^2, n = 1, 2, \ldots \); in that case \( X(x) \propto \cos nx \)
C) only when \( \lambda = -n^2, n = 1, 2, \ldots \); in that case \( X(x) \propto \sin nx \)
D) only when \( \lambda = -(n + \frac{1}{2})^2, n = 0, 1, 2, \ldots \); in that case \( X(x) \propto \cos (n + \frac{1}{2}) x \)
E) only when \( \lambda = -(n + \frac{1}{2})^2, n = 0, 1, 2, \ldots \); in that case \( X(x) \propto \sin (n + \frac{1}{2}) x \)
F) The \( \lambda < 0 \) case has no nonzero solutions.

\[ \lambda = -m^2 \Rightarrow X'' - m^2 X = 0 \quad \Rightarrow \quad \sum\omega(l) = A e^{m x} + B e^{-m x} \quad X'(x) = \int e^{m x} - B e^{-m x} \]

\[ \sum = e^m x \Rightarrow \sum \omega = 0 \quad \Rightarrow \quad \sum(\omega) = A \ast 0 \quad \Rightarrow \quad \sum(\pi) = A \ast 0 \quad \Rightarrow \quad \sum(\pi) = A \ast 0 \]

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3. For the case \( \lambda > 0 \), the problem (8) has nonzero solutions
A) only when \( \lambda = n^2, n = 1, 2, \ldots \); in that case \( X(x) = A e^{nx} + B e^{-nx} \)
B) only when \( \lambda = n^2, n = 1, 2, \ldots \); in that case \( X(x) \propto \cos nx \)
C) only when \( \lambda = n^2, n = 1, 2, \ldots \); in that case \( X(x) \propto \sin nx \)
D) only when \( \lambda = (n + \frac{1}{2})^2, n = 0, 1, 2, \ldots \); in that case \( X(x) \propto \cos (n + \frac{1}{2}) x \)
E) only when \( \lambda = (n + \frac{1}{2})^2, n = 0, 1, 2, \ldots \); in that case \( X(x) \propto \sin (n + \frac{1}{2}) x \)
F) The \( \lambda > 0 \) case has no nonzero solutions.

\[ \sum = e^{m x} \quad \Rightarrow \quad \sum(\omega) = A \cos mx + B \sin mx \quad X'(x) = 0 \quad \sum(\pi) = 0 \]

\[ \sum(\pi) = \left( n + \frac{1}{2} \right) \pi \]

In solving the two-point boundary value problem $X'' - 4X' + \lambda X = 0$, $X(0) = 0 = X(L)$, the cases that must be considered are:

A) $\lambda = 0$, $\lambda < 0$, $\lambda > 0$
B) $\lambda = 1$, $\lambda < 1$, $\lambda > 1$
C) $\lambda = 2$, $\lambda < 2$, $\lambda > 2$
D) $\lambda = 4$, $\lambda < 4$, $\lambda > 4$
E) All values of $\lambda$ can be considered simultaneously.

$$X(x) = e^{rx} + \sum_{n=1}^{\infty} (A_n e^{nx} + B_n e^{-nx}) \cos nx$$

Problems 5 - 7 concern the problem

$$u_{xx} + u_{yy} = 0, \quad u(x, 0) = f(x), \quad u(x, \pi) = g(x), \quad u_x(0, y) = 0 = u_x(\pi, y),$$

whose solution is of the form $u(x, y) = A_0 + B_0 y + \sum_{n=1}^{\infty} (A_n e^{n\pi y} + B_n e^{-n\pi y}) \cos nx$.

5. How should $f$ and $g$ be extended outside the interval $[0, \pi]$?
A) $f$ and $g$ should be extended as odd periodic functions with period $\pi$.
B) $f$ and $g$ should be extended as odd periodic functions with period $2\pi$.
C) $f$ and $g$ should be extended as even periodic functions with period $\pi$.
D) $f$ and $g$ should be extended as even periodic functions with period $2\pi$.

6. For $f(x) = \cos 2x$ and $g(x) = 3 \cos 4x$, what is true about the $A_n$ and $B_n$?
A) $A_2 = 1$, $A_4 = 3$, and all other $A_n$ and $B_n$ are zero.
B) $B_2 = 1$, $B_4 = 3$, and all other $A_n$ and $B_n$ are zero.
C) They are obtained by solving $\begin{cases} A_2 + B_2 = 1 \\ A_4 + B_4 = 3 \end{cases}$ and all other $A_n$ and $B_n$ are zero.
D) They are obtained by solving $\begin{cases} A_2 + B_2 = 1 \\ A_4 e^{4\pi} + B_4 e^{-4\pi} = 3 \end{cases}$ and all other $A_n$ and $B_n$ are zero.
E) They are obtained by solving $\begin{cases} A_2 e^{2\pi} + B_2 e^{2\pi} = 0 \\ A_4 e^{4\pi} + B_4 e^{-4\pi} = 3 \end{cases}$ and all other $A_n$ and $B_n$ are zero.

7. If we find that $B_2 = 1, A_3 = 5$, and all other $A_n$ and $B_n$ are zero, then the solution to problem (8) is
A) $\cos 2x + 5 \cos 3x$
B) $e^{-2y} \cos 2x + 5e^{3y} \cos 3x$
C) $(e^{2y} + e^{-2y}) \cos 2x + 5(e^{3y} + e^{-3y}) \cos 3x$
D) The solution cannot be determined from this information.
Sketch an approximation to the function \( f \). Label important points on your axes, and extend your drawing for several periods.

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{1000} a_n \cos n\pi x \quad \text{where} \quad a_n = -2 \int_{1/2}^{1} \cos(n\pi x) dx.
\]

\[
L = 1 \quad a_n = 2 \int_{0}^{1/2} f(x) \cos nx \, dx
\]

\[
\rho_{\phi} = \frac{\phi}{\lambda} \quad 2L = 2
\]

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Problems 9 - 11 deal with \( u_t = xu_{xx} + u_x \), \( u(0,t) = 0 = u(3,t), \) \( u(x,0) = 1 \). (#)

**9.** The two-point boundary value problem associated with (#) involves solving the ordinary differential equation

A) \( X'' + \lambda X = 0 \) \quad B) \( X'' + X' + \lambda X = 0 \) \quad C) \( X'' + X' + \lambda X = 0 \) \quad D) \( X'' + X' + \lambda = 0 \)

The variables cannot be separated for this problem.

\[
\overline{X'} = \begin{bmatrix} X'' \\ X' \end{bmatrix}, \quad \overline{X'} = X' \quad \Rightarrow \quad \overline{X}' = \begin{bmatrix} \lambda & X' \\ X' & \lambda \end{bmatrix} \begin{bmatrix} X' \\ X \end{bmatrix} = \lambda \overline{X}
\]

**10.** The boundary conditions for the two-point boundary value problem in 9) are

A) \( X(0) = 0 = X(3) \) \quad B) \( X(0) = 0 = X'(3) \) \quad C) \( X'(0) = 0 = X(3) \)

D) \( X'(0) = 0, \ X(0) = 1 \) \quad E) \( X'(0) = 0 = X(3), \ X(0) = 1 \)

**11.** The other ordinary differential equation that arises in solving (#) is

A) \( T' + \lambda T = 0 \) \quad B) \( T' - \lambda T = 0 \) \quad C) \( T' - \lambda = 0 \) \quad D) \( T' + \lambda = 0 \)

The variables still cannot be separated for this problem.

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12. The problem \( u_{tt} = u_{xx}, \ u(0,t) = 0 = u(\pi,t), \ u(x,0) = 0, \ u_t(x,0) = f(x) \) has a solution of the form \( u(x,t) = \sum_{n=1}^{\infty} (A_n \cos nt + B_n \sin nt) \sin nx. \) If \( f(x) \equiv 1, \) which of the following is NOT true?

A) \( B_n = \frac{2}{\pi} \int_{0}^{\pi} \sin nx \, dx \) \quad B) \( B_n = \frac{2}{\pi} (1 - (-1)^n) \) \quad C) \( B_n = \begin{cases} \frac{4}{n\pi} & \text{n odd} \\ 0 & \text{n even} \end{cases} \)

D) \( u(x,t) = \frac{4}{\pi} \left( \sin t \sin x + \frac{1}{3} \sin 3t \sin 3x + \frac{1}{5} \sin 5t \sin 5x + \ldots \right) \)

E) \( u(x,t) \) is the sum of even functions and is therefore even.

\[ u(x,t) = \sum_{n=1}^{\infty} A_n \sin nx \Rightarrow A_n = 0 \quad \Rightarrow \quad u(x,t) = 0 \]

\[ \int_{0}^{\pi} \sin nx \, dx = \left. \frac{\cos nx}{n} \right|_{0}^{\pi} = \frac{-1}{n} \left( (-1)^n - 1 \right) \]
13. A floating vibrating string can be modeled by the problem
\[ u_{tt} = u_{xx}, \; u_x(0, t) = 0 = u_x(\pi, t), \; u(x, 0) = f(x), \; u_t(x, 0) = g(x). \] (**)

The associated two-point boundary-value problem has nonzero solutions only when \( \lambda = n^2, \; n = 0, 1, 2, \ldots \), and the corresponding solutions \( X_n(x) \propto \cos nx \) (for \( n = 0, \; X_0 = \text{const.} \)). The associated equation \( T'' + n^2T = 0 \) has solutions

A) \( T_0(t) = A_n \cos nt + B_n \sin nt, \; n = 0, 1, 2, \ldots \)
B) \( T_0(t) = A_0/2, \; T_n(t) = A_n \cos nt + B_n \sin nt, \; n = 1, 2, \ldots \)
C) \( T_0(t) = A_0/2 + B_0t, \; T_n(t) = A_n \cos nt + B_n \sin nt, \; n = 1, 2, \ldots \)
D) \( T_0(t) = A_ne^{nt} + B_ne^{-nt}, \; n = 0, 1, 2, \ldots \)
E) \( T_0(t) = A_0/2, \; T_n(t) = A_ne^{nt} + B_ne^{-nt}, \; n = 1, 2, \ldots \)
F) \( T_0(t) = A_0/2 + B_0t, \; T_n(t) = A_ne^{nt} + B_ne^{-nt}, \; n = 1, 2, \ldots \)

14. For the problem
\[ u_{xx} + u_{yy} = 0, \; u_y(x, 0) = 0, \; u_y(x, \pi) = 0, \; u_x(0, y) = -1, \; u_x(\pi, y) = 1, \] (*)

the associated two-point boundary-value problem has solutions of the form \( Y_n(y) \propto \cos ny, \; n = 0, 1, 2, \ldots \) and the associated problem for \( X \) has a fundamental set of solutions consisting of \( \{e^{nx}, e^{-nx}\} \) for \( n = 1, 2, \ldots \) and \( \{1, x\} \) for \( n = 0 \). The solution to (*) is then of the form

A) \( \sum_{n=1}^{\infty} (e^{nx} + e^{-nx}) \cos nx \)
B) \( \sum_{n=1}^{\infty} (A_ne^{nx} + B_ne^{-nx}) \cos nx \)
C) \( 1 + x + \sum_{n=1}^{\infty} (e^{nx} + e^{-nx}) \cos nx \)
D) \( A_0 + B_0x + \sum_{n=1}^{\infty} (A_ne^{nx} + B_ne^{-nx}) \cos nx \)
E) \( \sum_{n=1}^{\infty} (A_ne^{nx} + B_ne^{-nx}) \cos nx + x \sum_{n=1}^{\infty} (C_ne^{nx} + D_ne^{-nx}) \cos nx \)

Problems 15 and 16 deal with the problem
\[ u_t = u_{xx}, \; u(0, t) = 1, \; u(\pi, t) = 2, \; u(x, 0) = 0 \] (*)

15. For large \( t \), the solution \( u(x, t) \) to the problem (*)

A) approaches a nonzero constant,  \quad B) goes to zero,  \quad C) goes to infinity
D) approaches the solution to the problem \( u_{xx} = 0, \; v(0) = 1, \; v(\pi) = 2, \)
E) cannot be determined from the information given.

16. Which of the following is NOT involved in solving problem (*)?
a) trying a solution of the form \( X(x)T(t) \)
b) solving the two-point boundary value problem \( X'' + \lambda X = 0, \; X(0) = 1, \; X(\pi) = 2 \)
c) solving \( T'' + n^2T = 0 \)
d) evaluating the expression \( \sum c_n X_n(x)T_n(t) \) at \( t = 0 \)