1. (10 points) Solve the initial-value problem \( y' + y^2 = 0, \quad y(0) = 1. \)

\[
\frac{dy}{dx} = -y^2 \\
\int \frac{dy}{y^2} = -x + c \\
-1 = 0 + c \\
-\frac{1}{y} = -x + c \\
-\frac{1}{y} = -x - 1 \\
\frac{1}{y} = x + 1 \\
y = \frac{1}{x + 1}
\]

2. (10 points) Find the general solution of \( y' + y + 1 = 0. \)

The integrating factor is \( e^x \)

\[
e^x y' + e^x y = -e^x \\
\frac{d}{dx}(e^x y) = -e^x \\
e^x y = -e^x + c \\
y = -1 + c e^{-x}
\]

\[y = -1 + c e^{-x}\]
3. (10 points) Find a real-valued fundamental set of solutions of $y'' + y = 0$.  

$$r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow y = e^{\pm it} = \cos t \pm i \sin t$$

$$\{y_1, y_2\} = \{\cos t, \sin t\}$$

4. (5 points) Find a particular solution of $y'' + y + 1 + t = 0$.

Try $y = at + b$

$$0 + at + b + 1 + t = 0$$

$$y' = a \quad \text{coeff of } t: \quad a + 1 = 0 \Rightarrow a = -1$$

$$y'' = 0 \quad \text{coeff of } 1: \quad b + 1 = 0 \Rightarrow b = -1$$

$$y = -t - 1$$

5. (5 points) Find the real-valued general solution of $y'' + y + 1 + t = 0$. (Hint: you have already done some of the necessary work above.)

$$y = c_1 \cos t + c_2 \sin t - t - 1$$
6. (10 points) The water in your 50-liter fish tank has become cloudy because there are many small particles suspended in the water. You decide to change the water by pouring in fresh water at a rate of 2 liters per minute and letting the (perfectly mixed) fluid drain out at the same rate. Write down a differential equation you could use to figure out how long it would take (and consequently how much water you would have to pour in) for the water in the tank to look clean. (Do not solve your equation.)

\[
\frac{dQ}{dt} = \text{volume of particles in water} - \frac{Q}{50} = 2
\]

\[
\frac{dQ}{dt} = -\frac{Q}{2s}
\]

7. (3 points each) Fill in the blank with the letter corresponding to the best description. Use the abbreviations SHM = simple harmonic motion; OD = overdamped; CD = critically damped; UD = underdamped; E = exponentially growing; B = beating; R = resonant; TSS = transient + steady state.

i) \( u'' + u + u = 0 \) \( r^2 + r + 1 = 0 \Rightarrow r = -1 \pm \sqrt{1 - 4} \) steady state

ii) \( \text{SHM} \) \( r^2 + 1 = 0 \Rightarrow r = \pm i \)

iii) \( R \) \( \ddot{u} + u = \cos t \)

iv) \( TSS \) \( \ddot{u} + \dot{u} + u = \cos t \)

v) \( O \) \( \ddot{u} + u = \cos 2t \)

vi) \( E \) \( \ddot{u} - \dot{u} - u = 0 \) not a spring-mass system!

vii) \( \text{SHM} \)

viii) \( UO \)

ix) \( R \)

x) \( O \)

xi) \( OD \)
8. (7 points) The solution of a certain spring-mass problem is \( u(t) = \cos 2\pi t - \sin 2\pi t \).

For this solution, the angular frequency is \( \frac{2\pi}{T} \), the period is \( \frac{1}{T} \),

\[
R \cos \left( 2\pi t + \phi \right) = \sqrt{R^2 \cos^2 \phi + \frac{1}{1 + (-1)^2}}
\]

and the amplitude is \( \sqrt{2} \).

9. For the equation \( \frac{dy}{dt} = y(y - 4) \):
   a) (3 points) sketch a graph of \( \frac{dy}{dt} \) as a function of \( y \).
   
   ![Graph of dy/dt vs y](image)

   b) (3 points) Identify the equilibrium solutions and determine the stability of each.

   \[
   y = 0 \quad \text{stable}
   
   y = 4 \quad \text{unstable}
   \]

c) (4 points) sketch a graph of \( y \) as a function of \( t \) for the initial condition \( y(0) = 2 \).

   ![Graph of y vs t](image)