Solutions to Midterm

1. (a) Dual to (Pb) is:

$$\begin{aligned} \max \quad & b^T y - u^T z \\ \text{s.t.} \quad & A^T y - z \le c \\ & z \ge 0 \end{aligned} \qquad (Db)$$

Take $y = 0$, $z_i = \max\{0, -c_i\}$, $i = 1, \dots, n$.

This is feasible.

(b) Consider the primal-dual pair of LPs:

$$\begin{aligned} \min \quad & -x_i \\ \text{s.t.} \quad & Ax = b \\ & x \ge 0 \end{aligned} \quad (P_i) \qquad\qquad \begin{aligned} \max \quad & b^T y \\ \text{s.t.} \quad & A^T y \le -e_i \end{aligned} \quad (D_i)$$

where $e_i = (0, \dots, 0, 1, 0, \dots, 0)^T$, the ith unit vector.

If x_i is bounded then (P_i) has finite optimal value, so (D_i) has finite optimal value and is feasible.

Let $\bar{y}$ be a feasible solution to (D_i)

Let $\hat{y}$ be a feasible solution to (D).

Then $\hat{y} + \alpha \bar{y}$ is feasible in (D) for any $\alpha \ge 0$,

since $A^T(\hat{y} + \alpha\bar{y}) \le A^T\hat{y} + \alpha A^T\bar{y} \le b - \alpha e_i \le b$

Also, let $s_\alpha = b - A^T(\hat{y} + \alpha\bar{y}) \ge \alpha e_i$.

As $\alpha \to \infty$, $(s_\alpha)_i \to \infty$. So s_i is unbounded in $\mathcal{Q}_D$.

2. (a) $BB^T = LQQ^TL^T = LL^T$

(b) $B^Ty = c$: $\quad Q^TL^Ty = c$

$$L^Ty = Qc$$

So: (i) Calculate Qc
(ii) Backsolve for y.

$Bd = a$: $\quad LQd = a$

So: (i) Forward solve for $w = Qd$
(ii) Calculate $d = Q^Tw$

(c) $B^Ty = c$: $\quad BB^Ty = Bc$

$$LL^Ty = Bc$$

So: (i) Calculate Bc
(ii) Forward substitute to find w satisfying $Lw = Bc$
(iii) Back substitute to find y.

$Bd = a$: $\quad BB^Tp = a$ with $B^Tp = d$

$$LL^Tp = a$$

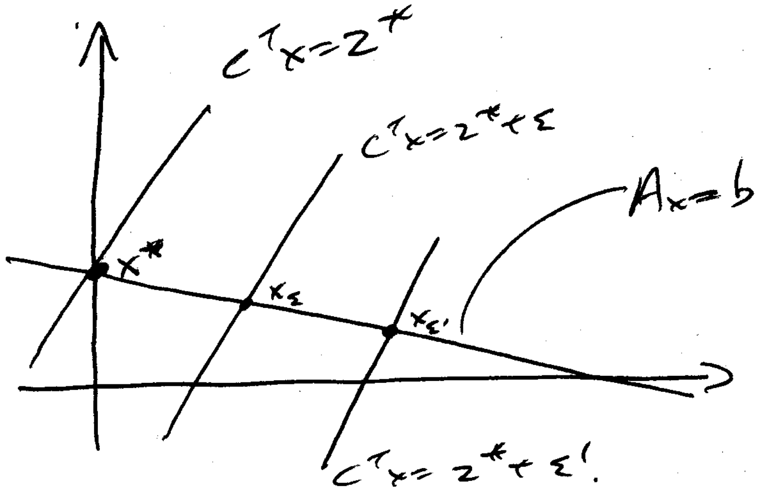
So (i) Find p using forward & back substitution
(ii) Calculate $d = B^Tp$.

(d) From the hint, we can assume there exist $\bar{L}, \bar{Q}$ with $\bar{B} = \bar{L}\bar{Q}$.

Then $\bar{L}\bar{L}^T = \bar{L}\bar{Q}\bar{Q}^T\bar{L}^T$ since $\bar{Q}$ orthogonal

$$\begin{aligned}
&= \bar{B}\bar{B}^T \\
&= (B+(a-b)e_p^T)(B+(a-b)e_p^T)^T = (B+(a-b)e_p^T)(B^T+e_p(a-b)^T) \\
&= BB^T + (a-b)b^T + b(a-b)^T + (a-b)(a-b)^T \quad \text{since } Be_p = b \text{ and } e_p^Te_p = 1 \\
&= LL^T + aa^T - bb^T \quad \text{since } BB^T = LL^T.
\end{aligned}$$

3.



Assume first $R_{\varepsilon'}$ is finite.

Let $x_{\varepsilon'} \in Q_{\varepsilon'}$ satisfy $e^T x_{\varepsilon'} = R_{\varepsilon'}$.

Let x^* solve (P).

Define $x_\varepsilon = x^* + \frac{\varepsilon}{\varepsilon'}(x_{\varepsilon'} - x^*)$ (1)

Then
$$c^T x_\varepsilon = c^T x^* + \frac{\varepsilon}{\varepsilon'} c^T (x_{\varepsilon'} - x^*)$$
$$= \left(1 - \frac{\varepsilon}{\varepsilon'}\right) c^T x^* + \frac{\varepsilon}{\varepsilon'} c^T x_{\varepsilon'}$$
$$\leq \left(1 - \frac{\varepsilon}{\varepsilon'}\right) c^T x^* + \frac{\varepsilon}{\varepsilon'} (c^T x^* + \varepsilon')$$
$$= c^T x^* + \varepsilon$$

So $x_\varepsilon \in Q_\varepsilon$, so $R_\varepsilon \geq e^T x_\varepsilon$.

Also, ~~necessarily, we have~~ from (1), we have
$$e^T x_\varepsilon = e^T x^* + \frac{\varepsilon}{\varepsilon'} e^T x_{\varepsilon'} - \frac{\varepsilon}{\varepsilon'} e^T x^*$$

So, $R_{\varepsilon'} = e^T x_{\varepsilon'} = \frac{\varepsilon'}{\varepsilon} e^T x_\varepsilon + \left(1 - \frac{\varepsilon'}{\varepsilon}\right) e^T x^*$
$$\leq \frac{\varepsilon'}{\varepsilon} R_\varepsilon \quad \text{since } e^T x_\varepsilon \leq R_\varepsilon,\ e^T x^* \geq 0,\ 1 - \frac{\varepsilon'}{\varepsilon} \leq 0.$$

4. Eg:

Supply = 5   Demand = 5   Supply = 10   Demand = 10

$$① \xrightarrow[\text{Cost} = 1]{} ② \xrightarrow[\text{Cost} = 100]{} ③ \xrightarrow[\text{Cost} = 1]{} ④$$

Optimal soln: $x_{12} = 5, \; x_{34} = 10, \; x_{23} = 0,$

cost $= 15.$

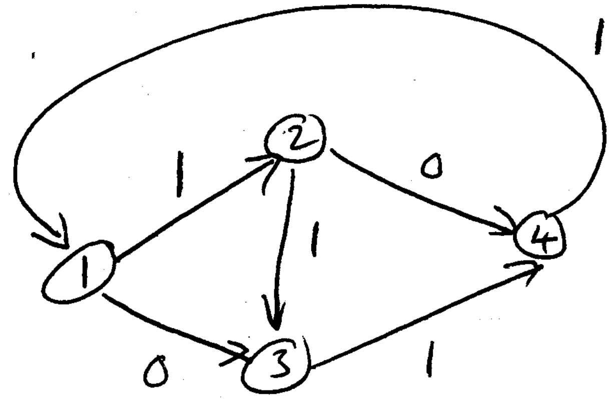
Reduce demand at node 2 to 4.

Reduce supply at node 3 to 9.

Optimal soln: $x_{12} = 5, \; x_{23} = 1, \; x_{34} = 10$

Cost $= 115.$

5.



Flows are shown.

Value: -11.

Dual constraints: $\pi_j - \pi_i - y_{ij} \leq c_{ij}$ for each arc (i,j).

y_{ij} are dual multipliers for upper bound constraints and must be nonnegative.

π_i are free.

For basic variables, $y_{ij} = 0$ and $\pi_j - \pi_i = c_{ij}$.

So:
$$\pi_2 - \pi_1 = 0$$
$$\pi_4 - \pi_3 = 0$$
$$\pi_1 - \pi_4 = -10$$

Take $\pi_1 = 0$. Then $\pi_2 = 0$, $\pi_4 = 10$, $\pi_3 = 10$.

For nonbasic variables:

$(1,3)$: at lower bound, so $y_{ij} = 0$.

Reduced cost: $c_{13} + \pi_1 - \pi_3 = -12$

$(2,3)$: at upper bound, so $c_{ij} = \pi_j - \pi_i - y_{ij}$

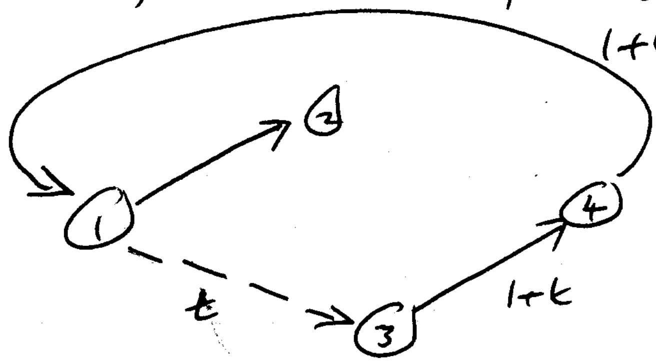
So: $y_{23} = \pi_3 - \pi_2 - c_{23} = +11$.

$(2,4)$: at lower bound, so $y_{24} = 0$

Reduced cost: $c_{24} + \pi_2 - \pi_4 = -12$.

So either x_{13} or x_{24} enters the basis.

Arbitrarily choose to bring x_{13} into basis:



Flow conservation requires this choice of flow.

Largest t before capacity constraints violated is $t = 0$. x_{34} leaves basis.

New bfs: $x_{12} = 1$ $\quad x_{13} = 0,$ $\quad x_{41} = 1$

Nonbasics: $x_{23} = 1$ $\quad x_{34} = 1$ $\quad x_{24} = 0.$

Find π_i from basic variables:

$$\pi_2 - \pi_1 = 0$$
$$\pi_3 - \pi_1 = -2$$
$$\pi_1 - \pi_4 = -10$$

Take $\pi_1 = 0$. Then $\pi_2 = 0$, $\pi_3 = -2$, $\pi_4 = 10$.

For nonbasic variables:

$(2,3)$: At upper bound, so $c_{ij} = \pi_j - \pi_i - y_{ij}$.

So $y_{23} = \pi_3 - \pi_2 - c_{23} = -1$

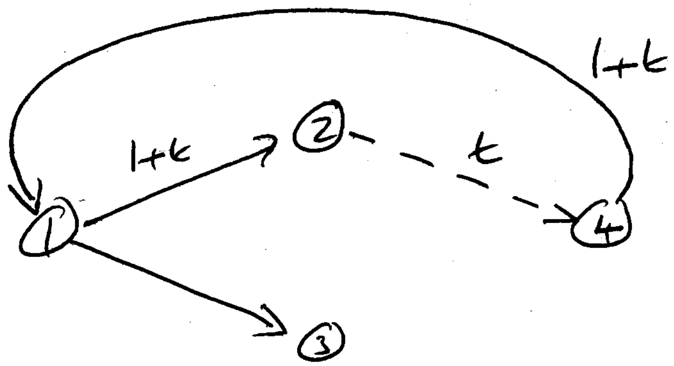
$(2,4)$: At lower bound, so $y_{24} = 0$.

Reduced cost: $c_{24} + \pi_2 - \pi_4 = -12$.

$(3,4)$: At upper bound, so $c_{ij} = \pi_j - \pi_i - y_{ij}$

So $y_{34} = \pi_4 - \pi_3 - c_{34} = 12$.

So choose to bring x_{24} into the basis:



x_{12} leaves basis, another degenerate pivot.

New bfs: basic: $x_{13} = 0,\ x_{24} = 0,\ x_{41} = 1.$

Nonbasics: $x_{12} = 1,\ x_{23} = 1,\ x_{34} = 1$

Find π_i from basic variables:
$$\pi_3 - \pi_1 = -2$$
$$\pi_1 - \pi_4 = -10$$
$$\pi_4 - \pi_2 = -2$$

Take $\pi_1 = 0$. Then $\pi_3 = -2,\ \pi_4 = 10,\ \pi_2 = 12.$

For nonbasic variables:

$(1,2)$: At upper bound, so $c_{ij} = \pi_j - \pi_i - y_{ij}$

So $y_{12} = \pi_2 - \pi_1 - c_{12} = 12$

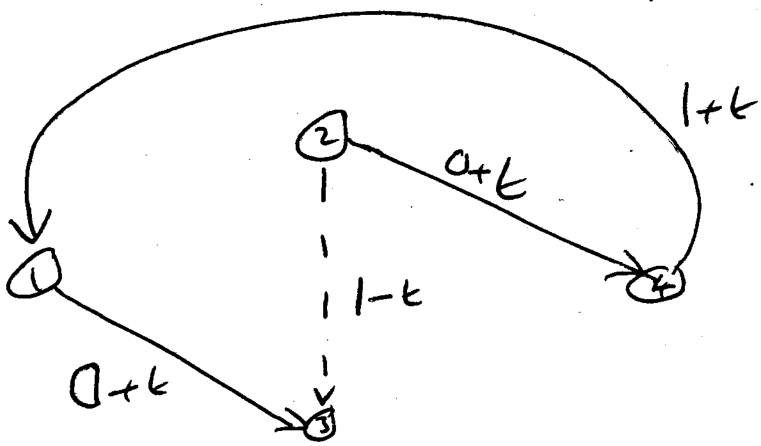
$(2,3)$: At upper bound, so $c_{23} = \pi_3 - \pi_2 - y_{23}$

So $y_{23} = \pi_3 - \pi_2 - c_{23} = -13$

$(3,4)$: At upper bound, so $c_{34} = \pi_4 - \pi_3 - y_{34}$

So $y_{34} = \pi_4 - \pi_3 - c_{34} = 12.$

x_{23} enters the basis, decreasing from its upper bound:



Get $t = 1$.

Choice for leaving variable: x_{13} leaves at upper bound.
x_{24} leaves at upper bound
x_{23} leaves at lower bound.

Choose third alternative, so x_{23} moves from its upper bound to its lower bound, and the basic sequence is unchanged.

New bfs: basics: $x_{13} = 1$, $x_{24} = 1$, $x_{41} = 2$

Nonbasics $x_{12} = 1$, $x_{23} = 0$, $x_{34} = 0$.

Find π_i from basic variables:

Eqns as before since basic sequence unchanged.
So $\pi_1 = 0$, $\pi_2 = 12$, $\pi_3 = -2$, $\pi_4 = 10$.

For nonbasic variables:
(1,2): π unchanged, x_{12} still at upper bound, so $y_{12} = 12$ still

(3,4): π unchanged, x_{34} still at upper bound, so $y_{34} = 12$ still

(2,3): Now at lower bound, so $y_{23} = 0$.

Reduced cost: $c_{23} + \pi_2 - \pi_3 = 13$.

All reduced costs nonnegative, so we are OPTIMAL.