1. Let $A$ be a set and $x$ a number. Show that $x$ is a limit point of $A$ if and only if there exists a sequence $x_1, x_2, \ldots$ of distinct points in $A$ that converges to $x$.

Proof. Let $x = \lim_{n \to \infty} x_n$. Then for any $\varepsilon > 0$ so that $|x - x_k| < \varepsilon$ for $k > m$. Thus, for any $\varepsilon > 0$ there exists $k \in A$ so that $|x - x_k| < \varepsilon$, which means that $x$ is a limit point (see Def. 3.2.1).

Let $x$ be a limit point of $A$. By the definition, for any $\varepsilon = \frac{1}{n}$, there exists $y \neq x$ so that $|y - x| < \frac{1}{n}$. Let's call $y = x_n$. The sequence $x_1, x_2, \ldots$ converges to $x$; there is a minor problem with it: the terms are not necessarily distinct. This flaw can be remedied by selecting a subsequence.

Set $x'_1 = x_1$. Take $E_1 = |x'_1 - x|$, and select $x'_2 = x_k$ where $|x_k - x| < E_1$. Then $x'_2 \neq x'_1$. Now take $E_2 = |x'_2 - x|$, and select $x'_k$ so that $|x_k - x| < E_2$. Set $x'_3 = x'_k$, and so on.

2. Prove that $K = \bigcap_{n=1}^{m} K_n$ and $C = \bigcap_{n=1}^{m} K_n$ are compact if all $K_n, K_2$ are compact.

Proof: Since each $K_n$ and $K_2$ is compact, it is bounded and closed (Thm 3.3.1). It means that any $k \in K_n$ lies between $a_n \leq k \leq b_n$. 

Therefore \( K = \bigcup_{n=1}^{m} K_n \) lies between \( \min_{n=1}^{m} a_n \) and \( \max_{n=1}^{m} b_n \).

i.e. \( K \) is bounded. By Thm 3.2.3, \( K \) is closed. Therefore \( K \) is compact.

Similarly, \( C = \bigcap K_i \) is Any of them. Therefore \( C \) is bounded. Again, by Thm 3.2.3 \( C \) is closed.

3. If \( B_1, ..., B_n \) is a finite open cover of a compact set \( A \), can the union \( B_1 \cup ... \cup B_n \) equal \( A \) exactly.

This is a trivial problem: The union is open (Thm 3.2.1) and therefore cannot be equal to a compact set which must be closed.

Comment: The only sets which both open and closed are \( \emptyset \) and \( \mathbb{R} \). Thus a compact set can be open only if it is empty.