1. Consider the initial value problem

\[(t-4)y'' + (\tan t)y' - 5y = \ln t, \quad y(t_0) = 2, \quad y'(t_0) = 0.\]

On what interval is this IVP guaranteed to have a unique solution when

(a) \(t_0 = 2\)? Explain.
(b) \(t_0 = 0\)? Explain.

2. (a) Verify that \(y_1 = x\) and \(y_2 = \ln t\) are solutions of the differential equation \(y'' + (y')^2 = 0\) on \(0 < t < \alpha\).
(b) Is \(c_1 y_1 + c_2 y_2\) \((c_1, c_2\) arbitrary constants\) a solution? How does this result relate to Thm. 3.2.2?

3. Prob. 25 on page 138 of the text. Also explain the answer in the book.

4. Let \(L\) be a linear differential operator.
   (a) If \(y = \phi(t)\) is a solution of \(L[y] = 0\), show that \(5\phi(t)\) is also a solution.
   (b) If \(y = \psi(t)\) is a solution of \(L[y] = g(t)\) (where \(g(t)\) is some given function), is \(5\psi(t)\) also a solution? Explain.
   (c) Of what equation is \(5\psi(t)\) a solution?

5. TURN THIS PAGE OVER.
5. In this problem we will prove uniqueness (that there is at most one solution) for the IVP
\[ ay'' + bx' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y'_0. \]
where \( y_0, y'_0, a > 0, b > 0, c > 0 \) are given constants and the given function \( g(t) \) is continuous on \( (-\infty, \infty) \).
(a) First we show that the only solution of the homogeneous IVP
\[ HIVP \quad ay'' + bx' + cy = 0, \quad y(0) = 0, \quad y'(0) = 0 \]
is \( y \equiv 0 \).
(b) Multiply the DE in (2) by \( y' \), integrate over \( 0 \leq s \leq t \) (with respect to \( s \)), and show that
\[ \frac{a}{2} [y'(t)]^2 + \frac{b}{2} \int_0^t [y'(s)]^2 ds + \frac{c}{2} [y(t)]^2 = 0 \]
for all \( t \).
(c) Show that Eq. (3) implies that \( y \equiv 0 \).
(d) If \( y = \phi_1(t) \) and \( y = \phi_2(t) \) are solutions of the IVP (1), show that, their difference \( w = \phi_1(t) - \phi_2(t) \) must be a solution of \( HIVP \) (2).

CONCLUSION: Hence, from part (b), the only solution of \( HIVP \) (2) is \( y \equiv 0 \), then
\[ \phi_1(t) - \phi_2(t) \equiv 0, \]
or
\[ \phi_1(t) \equiv \phi_2(t). \]
In other words, there cannot be two different solutions of IVP (1); there is at most one solution.

EXISTENCE? For many functions \( g(t) \), we know that a solution exists, for we can produce it (for example, by the Method of Undetermined Coefficients).