1(a) \( 3y'' - y' = 0 \) (1) is lin., homog., and has constant coefficients. Letting \( y = e^{at} \) give the
CHAR. EQ. \( 3\lambda^2 + \lambda = 0 \), \( \lambda(3\lambda + 1) = 0 \), \( \lambda = 0, -\frac{1}{3} \).

GEN'L SOLN \( y = c_1 e^{-t/3} + c_2 e^{6t} = [c_1 e^{-t/3} + c_2] \) \( \tag{2} \)
and \( y' = -\frac{c_1}{3} e^{-t/3} \)

IC's give \( y(0) = c_1 e^{-1/3} + c_2 = 0 \) \( \therefore c_1 = -6 e^{1/3} \)
\( y'(0) = -\frac{c_1}{3} e^{-1/3} = 2 \) \( \therefore c_2 = -c_1 e^{-1/3} = 6 \)

\( \therefore \) SOLN of IVP \( y = -6 e^{1/3} e^{-t/3} + 6 = 6 - 6 e^{-t-1/3} \)

(b) From (2), gen'l soln of DE (1), we see that, whatever values \( c_1 \) and \( c_2 \) have (determined by the IC's),
\[ y = c_1 e^{-t/3} + c_2 \rightarrow 0 + c_2 = c_2 \text{ as } t \rightarrow \infty. \]
\( c_2 \) is some number.

\( \therefore \) For no IC's will the solution become unbounded as \( t \rightarrow \infty \).

2. [Prob. 15, p. 157] \( 4y'' + 12y' + 9y = 0 \rightarrow 4\lambda^2 + 12\lambda + 9 = 0, \)
\( (2\lambda + 3)^2 = 0, \lambda = -\frac{3}{2}, -\frac{3}{2}. \)

GEN'L SOLN \( y = c_1 e^{-3t/2} + c_2 te^{-3t/2} \)
and \( y' = -\frac{3}{2} c_1 e^{-3t/2} + c_2 (1 - \frac{3}{2} t) e^{-3t/2} \)

(a) From IC's \( y(0) = 1, y'(0) = -4, \)
\[ y(0) = c_1 = 1, y'(0) = -\frac{3}{2} c_1 + c_2 = -4 \]
\( \therefore c_2 = -\frac{5}{2}. \)

\[ y = e^{-3t/2} - \frac{5}{2} te^{-3t/2} \]
in the sol'n of the IVP.

ROUGH GRAPH:

(b) Where does \( y(t) = 0 \)?
\[ y = e^{-3t/2} (1 - \frac{3}{2} t) = 0 \]
when \( 1 - \frac{3}{2} t = 0 \), \( t = \frac{2}{3} \). NOTE: \( e^{-3t/2} \) will never 0.

(c) Max. pt requires \( y' = 0 \). So we set
\[ y' = e^{3t/2} \left[ -\frac{3}{2} c_1 + c_2 (1 - \frac{3}{2} t) \right] = e^{3t/2} \left[ -\frac{3}{2} - \frac{5}{2} (1 - \frac{3}{2} t) \right] = 0 \]
\[ \therefore [-3 + \frac{5}{2} t_0] = 0, \ t_0 = \frac{6}{5}, y_{\text{max}} = -\frac{5}{2} e^{-3t/5} \]
[3] We've found only one pt. \((t_0, y_0)\) where \(y' = 0\). Curve starts at \(y = 1 > 0\) below \(t\)-axis, and approaches \(y = 0\) from below as \(t \to \infty\). So \((t_0, y_0)\) must be a min. pt.

(d) With \(I.C.'s: y(0) = 5, y'(0) = 0\), soln of IVP is found to be:
\[
y(t) = e^{-3t/2} \left[ 1 + (b + 3/2)t \right].
\]
- If \(b + 3/2 \geq 0\) \((b \geq -3/2)\), \(y(t) > 0\) for all \(t \geq 0\).
- If \(b + 3/2 < 0\) \((b < -3/2)\), then \([1 + (b + 3/2)t]\) becomes negative \& stays negative for \(t\) sufficiently large.
In either case \(e^{-3t/2}\) makes \(y(t) \to 0\) as \(t \to \infty\).

3. \(2y'' + 2y' + 5y = 0 \rightarrow \text{char. eq.: } 2n^2 + 2n + 5 = 0\)

\[
\lambda = \frac{-2 \pm \sqrt{4 - 4 \cdot 2 \cdot 5}}{4} = -1 \pm 3i
\]

G.S.: \(y = e^{-t/2} \left( c_1 \cos \frac{3\sqrt{2}}{2} t + c_2 \sin \frac{3\sqrt{2}}{2} t \right)\)

\[
y' = -\frac{t}{2} e^{-t/2} \left( c_1 \cos \frac{3\sqrt{2}}{2} t + c_2 \sin \frac{3\sqrt{2}}{2} t \right) + e^{-t/2} \left( -\frac{3}{2} c_1 \sin \frac{3\sqrt{2}}{2} t + \frac{3}{2} c_2 \cos \frac{3\sqrt{2}}{2} t \right)
\]

I.C.'s: \(y(0) = 5 \rightarrow c_1 = 5, \quad y'(0) = -1 \rightarrow -\frac{1}{2} c_1 + \frac{3}{2} c_2 = -1, \quad \text{and } c_2 = 1\)

Soln of IVP is:
\[
y = e^{-t/2} \left[ 5 \cos \frac{3\sqrt{2}}{2} t + \sin \frac{3\sqrt{2}}{2} t \right]
\]

"decaying oscillation"

4. \(y = c_1 e^t + c_2 e^{-3t/2}\). We work backwards. \(\lambda = 1, \lambda = -3/2\) are the roots of the char. eq., so \((\lambda - 1)(\lambda + 3/2) = \lambda^2 + \lambda - 3/4 = 0\) is the char. eq.

One DE is:
\[
y'' + \frac{t}{2} y' + \frac{3}{2} y = 0
\]

5. The general solution of the D.E. is \(y = c_1 e^{r_1 t} + c_2 e^{r_2 t}\)
where \(r_1, r_2 = (-b \pm \sqrt{b^2 - 4ac})/2a\) provided \(b^2 - 4ac > 0\).

In this case there are two possibilities. If \(b^2 - 4ac > 0\) then \((b^2 - 4ac)^{1/2} < b\) and \(r_1\) and \(r_2\) are real and

\(r_1, r_2 \to 0\), and hence \(y \to 0\), as \(t \to \infty\). If \(b^2 - 4ac < 0\) then \(r_1\) and \(r_2\) are complex conjugates with \(\text{negative}\) real part. Again \(e^{r_1 t} \to 0\) and \(e^{r_2 t} \to 0\); and hence \(y \to 0\), as \(t \to \infty\).

Finally, if \(b^2 - 4ac = 0\), then \(y = c_1 e^{r_1 t} + c_2 t e^{r_2 t}\) where \(r_1 = -b/2a < 0\). Hence, again \(y \to 0\) as \(t \to \infty\). This conclusion does not hold if either \(b = 0\) or \(c = 0\).