Readings: Shreve Sec. 4.5
Almgren, "Financial Derivatives and Partial Differential Equations"

Some history of the mathematics of finance:

Bachelier (1904): applying random walk models to price financial options, based on expected return.

Markowitz (1950s): theory for optimal portfolio investment, based on expected return and risk.

Black, Scholes, Merton (1970s): the right way to price financial options is to consider active hedging strategies and the value of equivalent portfolios.

We will consider how to assess the price of a European call option, which gives the holder (buyer) of the option the right, but not the obligation to purchase 1 share of an underlying asset from the writer (seller) of the option at a pre-agreed strike price $K$ at a future expiry time $T$.

This European call option has the following payoff, as a function of the asset price $x$ at the expiry time:

$$
\Lambda(x) = \max(x - K, 0) = (x - K)^+
$$

(For a European put option (option to sell rather than buy):

$$
\Lambda(x) = \max(K - x, 0) = (K - x)^+
$$

From a risk-neutral Markowitz (or just Bachelier) standpoint, one might say that the fair price to charge for a European call option is the expected payoff:

$$
\mathbb{E}\left[\Lambda(S(T))\right] = \mathbb{E}\left[(S(T) - K)^+\right]
$$

where $S(t)$ is the price of the underlying asset at time $t$, and the above expectation would be calculated by hypothesizing some sort of model for the asset price fluctuations (such as geometric Brownian motion):

$$
dS = \mu S dt + \sigma S dW(t)
$$

Markowitz incorporated considerations of risk, but it’s not entirely clear how to apply this to the general market.

The Black-Scholes-Merton approach gave a precise way to deal with the risk: by showing how to hedge it away. If one has written an option, then there is a strategy for actively trading the underlying asset to remove any risk to me from having written this option (under the idealizing
assumptions). One of the principles based on efficient markets is that no arbitrage opportunities should exist. Therefore the value of the riskless portfolio should be the same as the value of a riskless money market investment. We use geometric Brownian motion model for the asset price.

We define $V(x,t)$ to be the fair price of the European call option at time $t$ if the current value of the underlying asset price is $S(t) = x$.

We know that $V(x,T) = \wedge(x)$

We also can derive a partial differential equation by considering how the value of the option should evolve with time. (Eventually one would solve the resulting partial differential equation backwards in time.)

Ito lemma:

$$d V(S(t), t) = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial x} dS + \frac{1}{2} \frac{\partial^2 V}{\partial x^2} d\left[SS\right]$$

$$= \left(\frac{\partial V}{\partial t} (S(t), t) + \frac{1}{2} \sigma^2 S(t) \frac{\partial^2 V}{\partial x^2}\right) dt + \frac{\partial V}{\partial x} (S(t), t) dS(t)$$

The last term involves the uncertainty of the underlying asset price affecting the value of the option, which exposes me to risk. But Black-Scholes-Merton showed that this risk can be entirely removed by a replicating or compensating portfolio involving the underlying asset.

Imagine a self-financing portfolio of cash and the underlying asset:

- $C(t)$ is the value of cash held in the portfolio (money market investment)
- $D(t)$ is the number of shares of the underlying asset held in the portfolio

These need not be integers nor even positive numbers. Negative position in the asset is called a "short position," which is allowed.

Value of this cash plus asset portfolio:
$$\Pi(t) = D(t) S(t) + C(t)$$

One will take an active trading position by balancing the portfolio in response to changes in the underlying asset price:

$$D(t) = d(S(t), t)$$

$$C(t) = c(S(t), t)$$

$$d, c: \text{deterministic strategies}$$

How does this portfolio’s value evolve with time:

$$d\Pi = D(t) dS(t) + r C(t) dt$$

(no $dD$ term because of self-financing restriction: see Exercise 4.10 in Shreve.)

$r$ is the money market interest rate.

$$= d\left( S(t), t \right) dS(t) + r c(S(t), t) dt$$

Suppose I combine this portfolio with the European call option. What is the total value of that package?

$$d \left( V(S(t), t) + \Pi(t) \right)$$

$$= \left( \frac{\partial V}{\partial t}(S(t), t) + \frac{1}{2} \sigma^2 S^2(t) \frac{\partial^2 V}{\partial x^2} + r c(S(t), t) \right) dt$$

$$+ \left( \frac{\partial V}{\partial x}(S(t), t) + d(S(t), t) \right) dS$$

The risk term can be entirely eliminated through the following active hedge:

$$d\left( x, t \right) = -\frac{\partial V}{\partial x}(x, t)$$
This now makes the combination of the option, asset, and cash portfolio riskless!
This should be worth the same as a money market investment:

\[
d\left( V\left(S(t), t\right) + \Pi(t) \right) = r \left( V\left(S(t), t\right) + \Pi(t) \right) dt
\]

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rC = r \left( V\left(x, t\right) + d(x,t)X \right)
\]

Evaluate at \( X = S(t) \)

\[
\frac{\partial V}{\partial t} + r X \frac{\partial V}{\partial X} + \frac{1}{2} \sigma^2 X^2 \frac{\partial^2 V}{\partial X^2} = r \left( V - C \right)
\]

\[
V\left(x, T\right) = \Lambda\left(x\right)
\]

This is the Black-Scholes-Merton option pricing formula.