Name:

MATP6620/ISYE6770
Combinatorial Optimization and Integer Programming
Spring 2011

Midterm Exam, Tuesday, April 19, 2011.

Please do all four problems. Show all work. No books or calculators allowed. You may use any result from class, the homeworks, or the texts, except where stated. You may use one sheet of handwritten notes. The exam lasts 110 minutes.

SOLUTIONS

Q1	/20
Q2	/20
Q3	/20
Q4	/40
Total	/100

1. (20 points; each part is worth 10 points)

   (a) The binary variables x_1, x_2, x_3, and x_4 must satisfy the constraint

   $$6x_1 + 7x_2 + 4x_3 + 5x_4 \leq 13. \quad (*)$$

   Find a minimal cover inequality. Lift the cover inequality.

   (b) The binary variables z_1, z_2, z_3, and z_4 must satisfy the constraint

   $$6z_1 + 7z_2 - 4z_3 + 5z_4 \leq 9.$$

   Use the results of part (a) to derive an inequality that defines a facet of the set of $z \in \mathbb{B}^4$ satisfying the constraint. Why is your constraint facet-defining?

(a) $x_2 + x_3 + x_4 \leq 2$ (or $x_1 + x_3 + x_4 \leq 2$)

Lift $\max\ x_2 + x_3 + x_4$ s.t. $x_1 = 1$, $(*)$, x_i binary. max is 1

So lifted constraint is $x_1 + x_2 + x_3 + x_4 \leq 2$.

(b) Let $y_3 = 1 - z_3$, binary.

Write constraint as $6z_1 + 7z_2 + 4y_3 + 5z_4 \leq 9 + 4 = 13$.

From part (a), $z_1 + z_2 + y_3 + z_4 \leq 2$ is valid,

or $z_1 + z_2 + 1 - z_3 + z_4 \leq 2$

or $z_1 + z_2 - z_3 + z_4 \leq 1$.

Facet defining because minimal cover is facet defining over its set of variables, and we performed a maximal lifting on it.

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2. (20 points)

   The linearly dependent vectors $x^1, \ldots, x^m$ in $\mathbb{R}^n$ are all on the hyperplane $a^T x = b$, where $b \neq 0$. Show that they are affinely dependent.

Know $\sum_{i=1}^{m} \lambda_i x^i = 0$ for some λ_i not all zero.

So $0 = a^T\left(\sum_{i=1}^{m} \lambda_i x^i\right) = \sum_{i=1}^{m} \lambda_i a^T x^i = b \sum_{i=1}^{m} \lambda_i$,

so $\sum_{i=1}^{m} \lambda_i = 0$.

So, by definition, the vectors are affinely dependent.

3. (20 points)

Let $G = (V, E)$ be a graph with nonnegative edge weights w_e.

(a) (15 points) Show that the problem of finding the maximum weight matching on G is equivalent to finding a maximum weight *perfect* matching on another graph $G' = (V', E')$ with nonnegative edge weights w'_e, where V' and E' are obtained from V and E by adding certain vertices and edges.

(b) (5 points) How can you construct an equivalent *minimum* weight perfect matching problem, nonnegative.

(a) For each $v \in V$, introduce new vertex v', new edge (v, v') with weight 0.

Add additional edges so that the new vertices form a clique, and give all new edges weight 0.

Any matching M on G gives a perfect matching on G' of same weight: (i) if v is unmatched in M, use edge (v, v')
(ii) pair off the remaining unmatched vertices.

Conversely, given a perfect matching M' in G', keep only the edges in $M' \cap E$.

So solving max perfect matching problem on G' is equivalent to solving matching problem on G

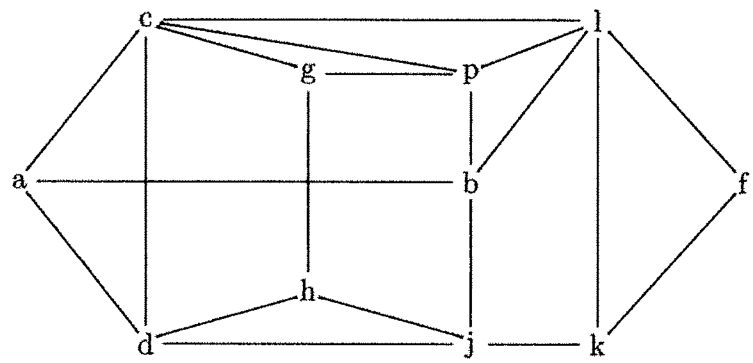
(b) Let $W = \max_{e \in E'} w_e$ Let $c_e = W - w_e$

Solve min weight perfect matching with c_e.

Get same solution since every perfect matching contains same number of edges.

4. (40 points; 6 parts on 3 pages)

This question is concerned with node packings in the following graph.



(a) (5 points) The set of feasible solutions to the node packing problem on a graph $G = (V, E)$ with n vertices and m edges can be written as

$$S := \{x \in \mathbb{B}^n : x_u + x_v \le 1 \; \forall (u, v) \in E\}.$$

We saw in class that the convex hull of S has dimension n and the nonnegativity constraints define facets of conv(S). Give another facet of conv(S) for the given graph.

(b) (5 points) Find a packing of cardinality 4 in the given graph.

(c) (10 points) Give a set of valid inequalitlies that together show that there cannot be a packing of cardinality greater than 4.

(a) Any clique, eg $x_a + x_c + x_d \le 1$.

(b) $\{a, f, g, j\}$

(c) $x_a + x_b \le 1$, $x_c + x_g + x_p \le 1$, $x_d + x_h + x_j \le 1$, $x_f + x_k + x_l \le 1$.

Add together. $\sum_{v \in V} x_v \le 4$.

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(d) (5 points) Is the graph perfect? (Justify your answer.)

(e) (7 points) Find nonnegative integral node weights c_v such that the optimal solution to the LP relaxation

$$\begin{array}{lrcl} \max & \sum_{v \in V} c_v x_v & & \\ \text{subject to} & x_u + x_v & \leq & 1 \quad \forall (u,v) \in E \\ & 0 \leq x_u & \leq & 1 \quad \forall v \in V \end{array}$$

is not integral.

(d) No: a-c-g-h-d-a is an odd hole.

(e) $c_a = c_c = c_g = c_h = c_d = 1$, all other $c_v = 0$.

Soln to LP relaxation is $x_a = x_c = x_g = x_h = x_d = \frac{1}{2}$.

(f) (8 points) The Lovasz Θ semidefinite programming relaxation of the node packing problem is

$$\begin{array}{lrcll} \max & \sum_{i=1}^n \sum_{j=1}^n X_{ij} & & & \\ \text{subject to} & X_{ij} & = & 0 & \text{if } (i,j) \in E \\ & \sum_{i=1}^n X_{ii} & = & 1 & \\ & X & \succeq & 0 & \end{array} \qquad (P)$$

which has dual problem

$$\begin{array}{lrcrcrcll} \min & & & z & & & & & \\ \text{subject to} & -S_{ii} & + & z & & & = & 1 & i = 1,\ldots,n \\ & -S_{ij} & & & + & y_{ij} & = & 1 & \text{if } (i,j) \in E \\ & -S_{ij} & & & & & = & 1 & \text{if } (i,j) \notin E \\ & S & & & & & \succeq & 0 & \end{array} \qquad (D)$$

Give a feasible solution to the dual problem with value 4.

Use the four cliques from part (c):

Set $z = 4$, $S_{ii} = 3 \ \forall i$

$y_{ab} = y_{cg} = y_{cp} = y_{gp} = y_{dh} = y_{dj} = y_{hj} = y_{fk} = y_{fl} = y_{kl} = 4,$

$y_{ij} = 0$ otherwise.

$$S = \begin{array}{c} a \\ b \\ c \\ g \\ p \\ d \\ h \\ j \\ f \\ k \\ l \end{array} \left[\begin{array}{cc|ccc|ccc|ccc} 3 & 3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 3 & 3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ \hline -1 & -1 & 3 & 3 & 3 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & 3 & 3 & 3 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & 3 & 3 & 3 & -1 & -1 & -1 & -1 & -1 & -1 \\ \hline -1 & -1 & -1 & -1 & -1 & 3 & 3 & 3 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & 3 & 3 & 3 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & 3 & 3 & 3 & -1 & -1 & -1 \\ \hline -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 3 & 3 & 3 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 3 & 3 & 3 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 3 & 3 & 3 \end{array}\right] \succeq 0,$$

since sum of block matrices $\begin{bmatrix} E & -E \\ -E & E \end{bmatrix}$, E: all ones, which are all psd.

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