Overview of Brownian Motors and Stochastic Models in Neuroscience

Tuesday, April 28, 2009
12:10 PM

The final exam is optional for everyone.

Absolute deadline for HW4 is Wednesday, May 6 at 5 PM.

Final exam, should you choose to take it, is Tuesday, May 5 at 3 PM. If you want to take the exam let me know by Friday, May 1 at 5 PM.

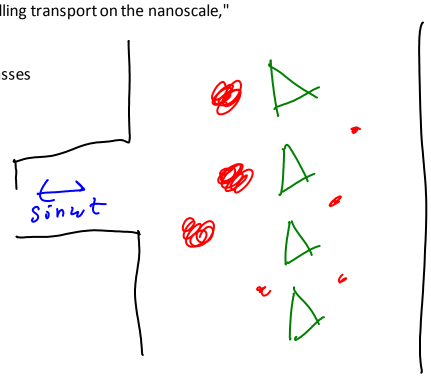
Office hours today are cancelled, but I will have office hours on Friday, May 1 at 4 PM. To meet at other times, contact me.

Course evaluations can be completed online, and the link dies after tomorrow.

If you are interested in an optional special topics lecture, please complete the form and return to me today by 3:30 PM.

Brownian motor applications in technology (Hanggi and Marchesoni "Artificial Brownian motors: Controlling transport on the nanoscale," 2009):

- nanopores
- sorters for DNA and particle masses



Many other technical applications: cold atoms in optical lattices, superconductors, quantum devices...

Introductory articles:

- Astumian & Hänggi, "Brownian motors," (2002)
- Reimann & Hänggi, "Introduction to the physics of Brownian motors," (2002)

Mathematical framework for computing drift and diffusion of Brownian motor as function of design parameters

- Pavliotis, "A multiscale approach to Brownian motors," (2005)

Stochastic Models in Neuroscience (Tuckwell, *Stochastic Processes in the Neurosciences*, Secs. 4.1-4.3)

Stochastic model for an individual neuron can be derived from fundamental equation for electrodynamics just like our stochastic models for Brownian motion, etc. could be derived from Newton's law. You view the neuron basically like a component of an electrical circuit. Incorporate a resistive element and a capacitance, but neglect inductances.

$$\frac{d\overbrace{Q}^{\text{charge}}}{dt} = \overbrace{I(t)}^{\text{current}}$$

$$Q = \underbrace{C}_{\text{capacitance (constant)}} \; \underbrace{V}_{\text{voltage inside neuron}}$$

$$C \frac{dV(t)}{dt} = I(t)$$

$$I(t) = I_{exc}(t) + I_{inh}(t) + I_{leak}(t)$$

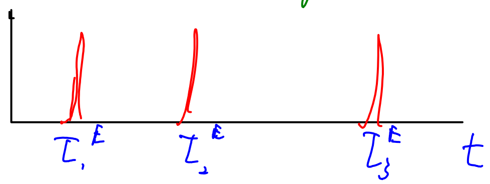
$$I_{leak}(t) = -g_L (V(t) - V_R)$$

leakage conductance $= \frac{1}{\text{resistance}}$

reset voltage (constant); can be thought of as extracellular voltage.

$$I_{exc}(t) = \sum_i a_E \, \delta(t - T_i^E)$$

$a_E \propto$ strength of excitatory input



At each reception of an excitatory spike, the voltage increases suddenly by an amount a_E / C

$$I_{inh}(t) = -\sum_i a_I \, \delta(t - t_i^I)$$

amplitude of inhibition

times of inhibitory input spikes

Every time an inhibitory spike is received, the voltage is suddenly decreased by an amount a_I / C

If one were simulating an entire network, then the times of excitation and inhibition would come from the solution of the dynamics of the other neurons whose axons are connected synaptically with the dendrites of the neuron under consideration. Howerver, it's often useful to consider a single neuron or a piece of a network that is receiving input signals from "outside" (another part of neural network or from the sensory organs).

Unless one has a good reason to the contrary, a widely used and applicable statistical model for the times of the spikes arising from outside the network is through a Poisson point process

- the number of spikes in nonoverlapping intervals are independent
- the number of spikes in a time interval of length t is given by a Poisson distribution with mean λt
- where λ is the intensity (or rate) of the spikes
- More precisely, we define a Poisson counting process $N(t)$ which is defined as the number of spikes received in the time interval $[0,t]$. The number of spikes received in a time interval $(s,t]$ is then just $N(t) - N(s)$.
- $$P(N(t) - N(s) = j) = e^{-\lambda t} \frac{(\lambda t)^j}{j!}$$

So in our neuron model we can take the input spike times as independent Poisson point processes for excitation and inhibition, with intensities (rates) λ_E, λ_I

and we associate to these spike times the Poisson counting processes

$N_E(t), \quad N_I(t)$

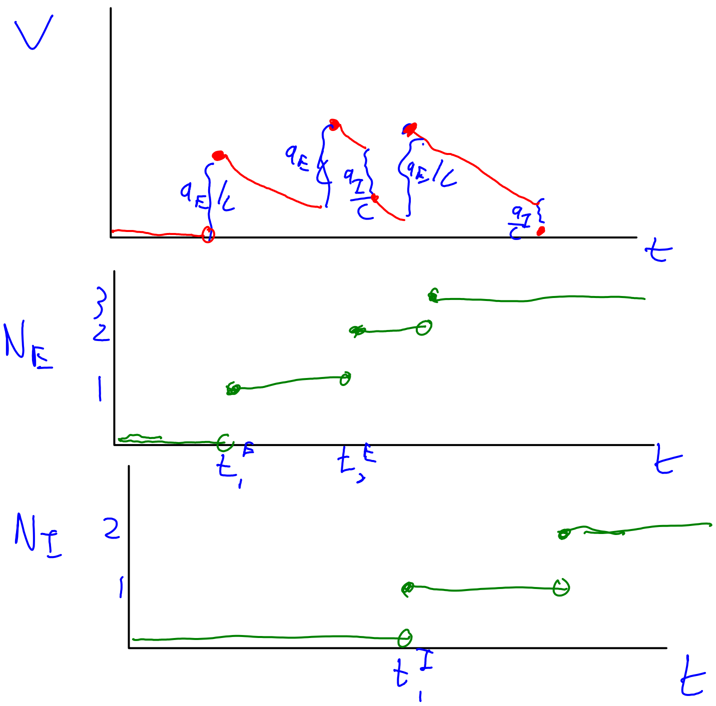
We can write the governing (current-driven) model for the neuron in physical form:

$$C \frac{dV(t)}{dt} = -g_L (V(t) - V_R) + \sum_i a_E \, \delta(t - T_i^E) - \sum a_I \, \delta(t - T_i^I)$$

$$-\sum_j q_I \, \delta(t-\tau_j^I) \Big)$$

Mathematical form that avoids delta functions:

$$C \, dV(t) = -g_L \left(V(t) - V_R\right) dt + q_E \, dN_E(t) - q_I \, dN_I(t)$$



One couples this stochastic "subthreshold dynamics" to the firing events when the voltage crosses threshold. So when the voltage crosses threshold, then it sends a spike and the voltage of the neuron that fired gets reset.

A fundamental question of interest is what are the properties of the times at which the neuron fires as a function of the input parameters (i.e., the currents arising from the input spikes.) In particular, how fast on average does the neuron fire as a function of design parameters?

Firing rate $\displaystyle \lim_{t\to\infty} \frac{N_0(t)}{t}$

$N_0(t) = \#$ spikes fired up to time t

Determining these properties relates to first-passage time calculations for the underlying stochastic dynamics. The calculations for this discontinuous spike model are rather complicated and unwieldy. consequently, one often uses a diffusion approximation to approximate the stochastic dynamics of the neuron voltage.

Start with Kolmogorov forward equatoin for the probabilty density for the neuron voltage under the subthreshold dynamics -- this is the analogue to the Fokker-

Planck equation for white noise forcing:

$$\frac{\partial \rho_V(v,t)}{\partial t} = \frac{\partial}{\partial v}\left(\frac{g_L}{c}(v-V_R)\rho_V(v,t)\right) + \lambda_E\left(\rho_V\left(v-\frac{q_E}{c},t\right)-\rho_V(v,t)\right) + \lambda_I\left(\rho_V\left(v+\frac{q_I}{c},t\right)-\rho_V(v,t)\right)$$

Under conditions in which the jumps are small (in some certain nondimensional sense), one can Taylor expand the right hand side to second order in the spike amplitudes to get:

$$\frac{\partial \rho_V(v,t)}{\partial t} = \frac{\partial}{\partial v}\left(\left(\frac{g_L}{c}(v-V_R) \underbrace{- \lambda_E\frac{q_E}{c} + \lambda_I\frac{q_I}{c}}_{= -\bar{I}_{ext}}\right)\rho_V(v,t)\right) + \frac{1}{2}\frac{\partial^2}{\partial v^2}\left(\underbrace{\left(\lambda_E\frac{q_E^2}{c^2} + \lambda_I\frac{q_I^2}{c^2}\right)}_{\sigma^2}\rho_V(v,t)\right)$$

+ higher order terms

This is FPE for

$$dV = \left(\bar{I}_{ext} - g_L(V-V_R)\right)dt + \sigma\, dW(t)$$