Reduction of Stochastic Fast-Slow Systems

Friday, April 10, 2009
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Again suppose we have a system of variables coupled together. The X variables correspond to large-scale variables or field values on a coarse scale and are of central interest. The Y variables correspond to more fine-scale information -- may not need to have explicit information about them for practical purposes other than the fact that they affect the variables X of interest.

$$d\vec{X} = \vec{a}(\vec{X},\vec{Y})\,dt$$

$$d\vec{Y} = \frac{1}{\varepsilon}\vec{b}(\vec{X},\vec{Y})\,dt + \frac{1}{\sqrt{\varepsilon}}\gamma(\vec{X},\vec{Y})\,d\vec{W}(t)$$

We would like to approximate this full coupled system by just a system involving the primary variables of interest, X; mathematical approaches can be brought to bear if these variables have some separation of time scales characterized by the small parameter $\varepsilon \ll 1.$

A few paradigmatic cases arise. The simplest situation can be approached by the (Standard) Method of Averaging

One can develop this systematically using the perturbation expansion approach from last time (though in case one stops after two rather than three equations) but we'll just explain the result intuitively.

Since the Y variable varies on a fast time scale relative to X; we can treat the Y variables as always having a quasi-stationary probability distribution corresponding to a fixed value of the variable X=x.

That is, if we look at the conditional probability density for Y given X, then after a little work one can show that if we freeze X=x and allow Y to evolve, then this conditional probability density satisfies the Fokker-Planck equation:

$$\frac{\partial}{\partial t} p_{\vec{Y}|\vec{X}}(\vec{y}|\vec{x};t) = \frac{1}{\varepsilon}\mathcal{L}_{\vec{Y}}\, p_{\vec{Y}|\vec{X}}(\vec{y}|\vec{x};t)$$

$$= \frac{1}{\varepsilon}\vec{\nabla}_y\cdot\left(\vec{b}\,p_{\vec{Y}|\vec{X}}\right) + \frac{1}{2\varepsilon}\vec{\nabla}_y\vec{\nabla}_y : \left(\gamma\gamma^T p_{\vec{Y}|\vec{X}}\right)$$

where $$P\left(\vec{Y}(t)\in B \,\middle|\, \vec{X}=\vec{x}\right) = \int_B p_{\vec{Y}|\vec{X}}(\vec{y}|\vec{x};t)\,d\vec{y}$$

Then on a time scale long compared to the time scale of Y but short compared to the time scale of X, the conditional probability density will quickly reach a stationary distribution conditioned on X.

For $\varepsilon \ll t \ll 1$

$$p_{\vec Y|\vec X}(\vec y|\vec x;t) \to p^{(s)}_{\vec Y|\vec X}(\vec y|\vec x)$$

where $$\mathcal{L}_{Y}\, p^{(s)}_{\vec Y|\vec X}(\vec y|\vec x) = 0$$

$$\vec\nabla_y \cdot \left(\vec b\, p^{(s)}_{\vec Y|\vec X}\right) + \frac{1}{2}\vec\nabla_y\vec\nabla_y : \left(\gamma\gamma^T p^{(s)}_{\vec Y|\vec X}\right) = 0$$

Along with conditions of decay at infinity, this is a second order elliptic equation and one can generically show this usually has a unique solution which has integral =1, provided the system is ergodic, meaning that the statistics of the state of a single trajectory essentially samples all possible states with the same probability distribution as observing a large number of independent systems in the statistically stationary state. (Time averages = ensemble averages).

Then for $\varepsilon \ll 1$,

$$\vec X(t) = \vec X^\#(t) + O(\varepsilon^{1/2})$$

$$d\vec X^\#(t) = \vec a^\#(\vec X(t))\,dt$$

where

$$\vec a^\#(\vec x) = \int \vec a(\vec x,\vec y)\, p^{(s)}_{\vec Y|\vec X}(\vec y|\vec x)\, d\vec y$$

This is the most intuitive result -- the coefficient in the equation for the slow variable is just averaged, for each x, over all possible values of y according to the probability distribution they would have when the slow variable X=x.

Two fundamental physical examples:

What would this look like for our Brownian motion example:

$\vec Y = \vec V'$ $\quad$ $\vec X = \vec X'$

$(d\vec X = \vec V\, dt)$

$$\vec a(\vec x,\vec y) = \vec y = \vec v'$$

$$p^{(s)}_{\vec Y|\vec X}(\vec y|\vec x) = p^{(s)}_{\vec V|\vec X}(\vec v'|\vec x')$$

$$= p^{(s)}_{\vec V}(\vec v')$$

$$= \frac{e^{-\frac12|\vec v'|^2}}{\sqrt{2\pi}}$$

(in this case $\vec V, \vec X$ independent in thermal equil

$$p^{(s)}(\vec x',\vec v') = \frac{e^{-\frac12|\vec v'|^2 - \mu\Phi(\vec x')}}{Z}$$

$$\vec{a}^{\#}(\vec{x},\vec{y}) = \vec{a}^{\#}(\vec{x}', \vec{v}') = \int_{\mathbb{R}^d} \vec{v}'\, e^{-\frac{1}{2}|\vec{v}'|^2/k_B T}\, d\vec{v}'$$

$$= \vec{0}$$

This is what we would have gotten if we had taken our reference time scale to be $T_a = \dfrac{l_F}{\sqrt{\frac{k_B T}{m}}}$

velocity scale

advective

$$t' = \frac{t}{T_a}$$

$$d\vec{X}' = \vec{V}'\, dt'$$

$$d\vec{V}' = \left(-\frac{1}{\varepsilon}\vec{V}' - \frac{1}{\varepsilon}\mu\vec{\nabla}\Phi\right)dt' + \frac{1}{\sqrt{\varepsilon}}\, d\vec{W}(t')$$

This system, if we had applied the perturbation theory, which is equivalent to the method of averaging, would have told us that

$$\vec{X}'(t) \sim \vec{X}^{\#}(t) + O(\sqrt{\varepsilon})$$

$$d\vec{X}^{\#}(t) = 0$$

$$\vec{X}'(t) \sim O(\sqrt{\varepsilon})$$

The conclusion from this calculation would have been that we are looking at the system on a time scale which is not an interesting one for the position. It's not doing too much (yet). What we did is actually use a different reference time scale that anticipated the diffusive character of the motion and gave us an interesting result (and a more complicated calculation).

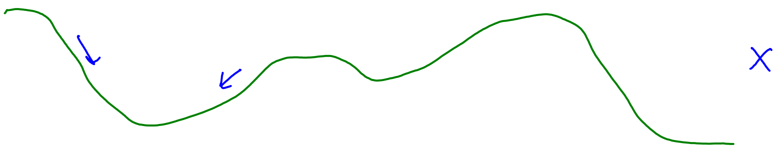
An important canonical example from physics

Consider a system given by overdamped dynamics governed by some potential energy of the state variables $\Phi(\vec{x}, \vec{y})$

$$d\vec{X} = -\vec{\nabla}_x \Phi\, dt + \sigma\, d\vec{W}_x(t)$$

$$d\vec{Y} = -\frac{1}{\varepsilon}\vec{\nabla}_y \Phi\, dt + \frac{\sigma}{\sqrt{\varepsilon}}\, d\vec{W}_y(t)$$

This is after nondimensionalization; σ is a nondimensional quantity proportional to square root of temperature.



The standard set up for this: X are a few collective variables wheras Y are some more detailed variables.

Examples:

X	Y
magnetization	spin variables
electromagnetic wave amplitude	atomic states
mean flow, vorticity	small-scale fluctuations
position of solute particle	positions of solvent/fluid particles

(aka order parameter)

If we have a time scale separation between the collective variables and the fine scale variables,then applying the standard averaging procedure, we first have to solve for the conditional probability density for the fine variables, given a particular value for the collective particle.

$$0 = \mathcal{L}_Y \, p^{(s)}_{Y|X}(\vec{y}\,|\,\vec{x})$$

$$= \vec{\nabla}_y \cdot \left(\vec{\nabla}_y \Phi \, p_{Y|X} \right) + \frac{1}{2} \vec{\nabla}_y \vec{\nabla}_y : \left(\sigma^2 p_{Y|X} \right)$$

Solve by integrating:

$$0 = \vec{\nabla}_y \cdot \left(\vec{\nabla}_y \Phi \, p^{(s)}_{Y|X} + \frac{1}{2} \sigma^2 \vec{\nabla}_y \, p^{(s)}_{Y|X} \right)$$

$$\vec{\nabla}_y \Phi \, p^{(s)}_{Y|X} + \frac{1}{2} \sigma^2 \vec{\nabla}_y \, p^{(s)}_{Y|X} = \vec{c}$$

$\vec{c} = \vec{0}$ to get normalizable PDF

Solve by integrating factor

$$p^{(0)}_{Y|X}(\vec{y}\,|\,\vec{x}) = \frac{e^{-2\Phi(\vec{x},\vec{y})/\sigma^2}}{Z_{\vec{x}}}$$

$$Z_{\vec{x}} = \int e^{-2\Phi(\vec{x},\vec{y})/\sigma^2}\, d\vec{y}$$

(In dimensional temrs, this is just the Gibbs-Boltzmann ditsribution for y when the variable X=x is frozen:

$$\frac{e^{-\Phi(\vec{x},\vec{y})/k_B T}}{Z_x}$$

The averaged equation for the collective variable is then:

$$\vec{X}(t) \sim \vec{X}^{\#}(t) + O(\sqrt{\varepsilon})$$

$$d\vec{X}^{\#}(t) = \vec{a}^{\#}(\vec{X}^{\#}(t))$$

$$\vec{a}^{\#} = \int \underbrace{\left(-\vec{\nabla}_x \Phi(\vec{x},\vec{y})\right)}_{\vec{a}(\vec{x},\vec{y})} \underbrace{\frac{e^{-2\Phi(\vec{x},\vec{y})/\sigma^2}}{Z_x}}_{p_{Y|X}(\vec{y}|\vec{x})}\, d\vec{y}$$

$$\vec{a}^{\#} = -\nabla F'(\vec{x})$$

$$F'(\vec{x}) = -\frac{\sigma^2}{2} \ln \int e^{-2\Phi(\vec{x},\vec{y})/\sigma^2}\, d\vec{y}$$

$$= -\frac{\sigma^2}{2} \ln Z_x$$

And if we returned to dimensional variables, this takes on a familiar form:

$$F(\vec{x}) = -k_B T \ln Z_x = \text{free energy for fixed value of } \vec{x}$$

$$Z_x = \int e^{-\Phi(\vec{x},\vec{y})/k_B T}\, d\vec{y}$$

$$d\vec{X}^{\#}(t) = -\vec{\nabla}_x F(\vec{X}(t)) + \sqrt{\sigma}\, d\vec{W}(t)$$

A few more remarks on going beyond of method of averaging.

One can not only derive an effective deterministic averaged equation for the slow variables; one can also derive a simplified equation for the behaviorof their fluctuations to leading order via an effective stochastic differential equation. The formulas and arguments work similarly as discussed above -- see Arnold, "Hasselmann's Program Revisited: The Analysis of Stochasticity in Deterministic Climate Models"

But recall from our first example that the method of averaging can be boring if the averaged coefficient is zero. In this case, one needs to rescale the problem to a longer time scale, just as we did for the Brownian motion problem:

After you do this, the equations in the new nondimensionalization will look like:

$$d\vec{X} = \frac{1}{\epsilon}\, \vec{a}(\vec{X},\vec{Y})\, dt$$

$$d\vec{Y} = \frac{1}{\epsilon^2}\, \vec{b}(\vec{X},\vec{Y})\, dt + \frac{1}{\epsilon}\, \mathcal{L}(\vec{X},\vec{Y})\, d\vec{W}(t)$$

One can check this by looking at our Fokker-Planck equation (Klein-Kramer equation) for the Brownian motion problem and writing down the corresponding SDE.

$$d\vec{X}' = \frac{1}{\epsilon}\left(\vec{V}' - \mu \vec{f}(\vec{X}')\right) dt'$$

$$d\vec{V}' = -\frac{1}{\epsilon^2}\, \vec{V}'\, dt' - \frac{1}{\epsilon}\, d\vec{W}(t')$$