Perturbation Theory for Stochastic Mode Reduction

Tuesday, March 31, 2009
12:08 PM

Homework 3 posted, due Friday, April 10 at 12 PM.

Now we will exploit the smallness of one of the governing parameters.

We claim that for many applications in microphysics, $\varepsilon \ll 1$

Why?

$$\varepsilon = \frac{\sqrt{k_B T m}}{l_F \, \gamma}$$

$$k_B = 1.3 \times 10^{-16} \ \frac{g\, cm^2}{s^2 K}$$

$$T = 300 \ K$$

$$k_B T \approx 4 \times 10^{-14} \ g \frac{cm^2}{s^2}$$

$$m \approx \frac{4\pi}{3} \rho a^3 \approx 3 \left(1 \frac{g}{cm^3} \right) a^3 \approx 3 \, g \left(\frac{a}{cm} \right)^3$$

size of particle

$$\begin{aligned} \gamma &\approx 6\pi \rho \nu a \\ &\approx (20)\left(1 \frac{g}{cm^3}\right)\left(.01 \ \frac{cm^2}{s}\right) a \\ &= .2 \ \frac{g}{s} \left(\frac{a}{cm}\right) \end{aligned}$$

$$\varepsilon = \frac{\sqrt{4 \times 10^{-14} \, g \frac{cm^2}{s^2} \ 3 g \left(\frac{a}{cm}\right)^3}}{l_F \ .2 \frac{g}{s} \left(\frac{a}{cm}\right)}$$

$$\varepsilon = 2 \times 10^{-6} \frac{\sqrt{\frac{a}{cm}}}{\frac{l_F}{cm}}$$

particle size

forcing length scale

Typical: $a \sim 10^{-4}$ cm

$l_F \sim 10^{-4}$ cm

$\ell_F \sim 10^{-4}$ cm

$$\varepsilon = 2 \times 10^{-6} \frac{\sqrt{10^{-4}}}{10^{-4}} = 2 \times 10^{-4} \ll 1$$

In microphysics, the forcing length scale will typically be micron size or larger, and the particle size will typically be micron size or smaller, which only makes the parameter ε even smaller than this.

This is fine and good mathematically, and we can see that the parameter ε roughly measures how strong friction is, how small the mass is in some nondimenisonalized sense, but can we explain more precisely what this combination of parameters means? A fruitful way to understand what nondimensional parameters mean is often to think about whether they express some notion of a ratio of two time scales or a ratio of two length scales associated to the physics.

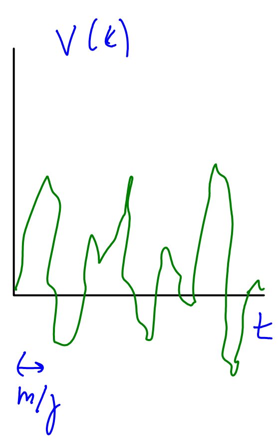
In this case

$$\varepsilon = \sqrt{\frac{m/\gamma}{\gamma \ell_F^2/k_B T}} = \sqrt{\frac{m/\gamma}{\ell_F^2/D}}$$

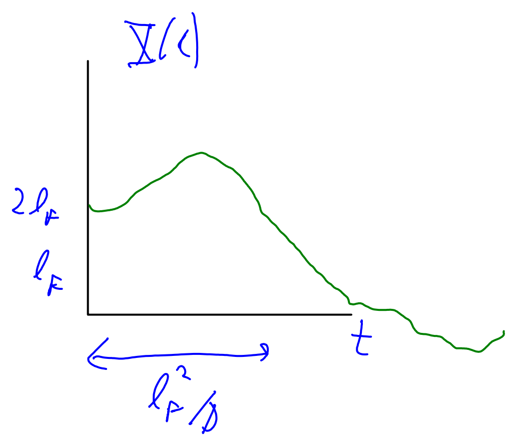
m/γ = momentum/velocity relaxation time scale = time scale for $\vec{V}$

ℓ_F^2/D → diffusion time scale; time for particle to freely diffuse over the reference length scale $= T$ = time scale for position $\vec{X}$

With this interpretation, we see that systems with small ε are ones for which the time scale of the momentum/velocity is short compared to the time scale on which the position changes.



fast variable



slow variable

The intuition behind stochastic mode reduction is that complex systems involving fast and slow variables can sometimes be simplified using techniques of multiple scale analysis into reduced systems only involving the slow variable, but taking into account the effects the fast variable has on the slow variable by somehow "averaging" the effect of the fast variables. This is really useful because systems with fast and slow variables are numerically stiff - this leads into multiscale computing -- later lecture.

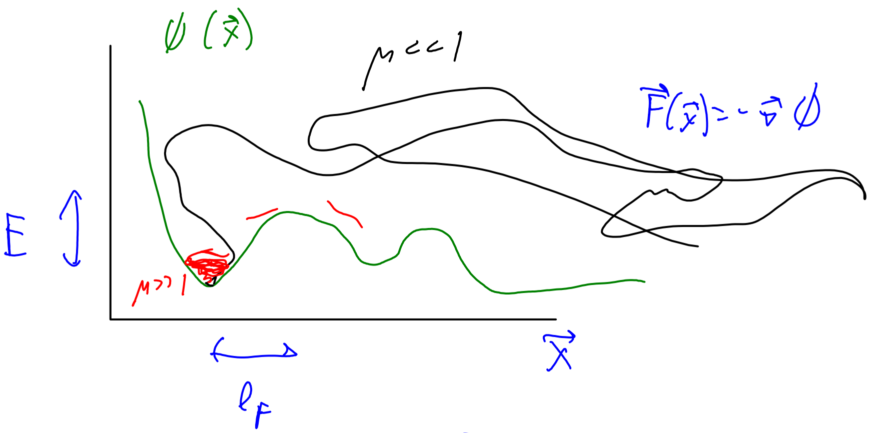
We've said that the parameter ε is small, what about μ?

$$\mu = \frac{\ell_F \bar{F}}{k_B T}$$

$\ell_F \bar{F}$ ~ energy scale of applied force

$k_B T$ = thermal kinetic energy of particle

We can interpret this as a ratio of energies.



Slope of potential energy gives force, so $\bar{F} \sim \dfrac{E}{\ell_F}$

The size of μ depends on the application of interest, whether thermal kinetic energy or potential energy variations dominate the dynamics. Typically if we are looking at a system useful from a biological or physical point of view and we're worrying about thermal fluctuations, this ratio is not too different from order unity.

So for this reason we shall take $\mu \sim O(1)$

Undergraduate colloquium (but grad students welcome) today at 2PM, Amos Eaton 214

Nina Fefferman, Introduction to modeling epidemiology and sociobiology on networks

So we will now proceed with developing a perturbation theory based on the assumption of $\varepsilon << 1, \quad \mu \sim O(1)$

Here is where a good choice of reference units is going to make this calculation sensible. When you do perturbation theory with an explicit small parameter, then every other parameter and variable in the system should be order unity (neither very small nor very large) or your answer will not be relevant.

Perturbation theory: Holmes, Introduction to Perturbation Methods
Lin & Segel, Mathemtaical Methods Applied to Determinitsic Problems in the Natural Sciences

What we will do is a form of singular perturbation theory without going into full-blown multiple scales analysis (which you would need to do in general, particularly if you want to connect initial data).

Perturbation theory (at its simplest) starts with the premise that all quantities depending on a small parameter can be expanded in a power series (asymptotic series) with respect to the small parameter. Kind of Taylor series but you don't care about the whole series converging because you never compute more than a few terms.

$$p'_{\mathbf{X},V}(\vec{x}',\vec{v}',t;\varepsilon) = p'^{(0)}_{\mathbf{X},\mathbf{V}}(\vec{x}',\vec{v}',t) + \varepsilon\, p'^{(1)}_{\mathbf{X},\mathbf{V}}(\vec{x}',\vec{v}',t) + \varepsilon^2 p'^{(2)}_{\mathbf{X},V}(\vec{x}',\vec{v}',t) + O(\varepsilon^3)$$

To organize the perturbation theory, it helps to write the equation in symbolic terms:

$$(\varepsilon^{-2}\mathcal{L}_0 + \varepsilon^{-1}\mathcal{L}_1 + \mathcal{L}_2)\, p'_{\mathbf{X},\mathbf{V}}(\vec{x}',\vec{v}',t) = 0$$

operators

$$\mathcal{L}_0 g = -\vec{\nabla}_{v'}\cdot(\vec{v}' g) - \Delta_{v'} g$$

$$\mathcal{L}_1 g = \vec{\nabla}_{x'}\cdot(\vec{v}' g) + \mu \vec{\nabla}_{v'}\cdot(\vec{\nabla}\varphi\, g)$$

$$\mathcal{L}_2 g = \frac{\partial g}{\partial t'}$$

keep track of formal error

Now plug in the perturbation ansatz into the equation:

$$(\varepsilon^{-2}\mathcal{L}_0 + \varepsilon^{-1}\mathcal{L}_1 + \mathcal{L}_2)\left(p'^{(0)}_{\mathbf{X},V} + \varepsilon\, p'^{(1)}_{\mathbf{X},V} + \varepsilon^2 p'^{(2)}_{\mathbf{X},\mathbf{V}} + O(\varepsilon^3)\right) = 0$$

Expand + organize from most important to least important terms.

$$\varepsilon^{-2}\left(\mathcal{L}_0 p'^{(0)}_{\mathbf{X},\mathbf{V}}\right) + \varepsilon^{-1}\left(\mathcal{L}_1 p'^{(0)}_{\mathbf{X},\mathbf{V}} + \mathcal{L}_0 p'^{(1)}_{\mathbf{X},\mathbf{V}}\right) + \left(\mathcal{L}_2 p'^{(0)}_{\mathbf{X},\mathbf{V}} + \mathcal{L}_1 p'^{(1)}_{\mathbf{X},\mathbf{V}} + \mathcal{L}_0 p'^{(2)}_{\mathbf{X},\mathbf{V}}\right) + O(\varepsilon) = 0$$

Each term must vanish separately, order by order, and we usually think about this as occurring sequentially starting with the most important term:

Perturbation hierarchy or Asymptotic hierarchy

$$O(\varepsilon^{-2}): \quad \mathcal{L}_0 p'^{(0)}_{\mathbf{X},\mathbf{V}} = 0$$

$$O(\varepsilon^{-1}): \quad \mathcal{L}_1 p'^{(0)} + \mathcal{L}_0 p'^{(1)} = 0$$

$$O(\varepsilon^{-1}): \quad \mathcal{L}_1 \rho'^{(0)}_{X,V} + \mathcal{L}_0 \rho'^{(1)}_{X,V} = 0$$

$$O(1): \quad \mathcal{L}_2 \rho'^{(0)}_{X,V} + \mathcal{L}_1 \rho'^{(1)}_{X,V} + \mathcal{L}_0 \rho'^{(2)}_{X,V} = 0$$

The argument is that all the expressions in parentheses are assumed independent of ε and then if these terms didn't vanish, the most important one would have nothing to balance it for sufficiently small values of ε therefore the most important coefficient is zero, now repeat argument for next most important coefficient, etc.

Goal for this case will be to determine what the dynamics of the leading order solution $\rho'^{(0)}_{X,V}$ is, and it should be simpler than the original equation because it should not involve ε.

But we'll see that to do this, we'll actually need to go through second order in the perturbation theory (which is what happens for singular problems).

The procedure for constructing this approximate solution is to solve the equations in the perturbation hierarchy order by order, starting with the most important, until you have the desired information (which in our case is just the leading order solution).

Let's just begin by solving the equation with the most importance (highest amplitude):

$$O(\varepsilon^{-2}): \quad \mathcal{L}_0 \rho'^{(0)}_{X,V} = 0$$

$$-\vec{\nabla}_{v'} \cdot \left(\vec{v}' \rho'^{(0)}_{X,V} \right) - \Delta_{v'} \rho'^{(0)}_{X,V} = 0$$

$$-\vec{\nabla}_{v'} \cdot \left(\vec{v}' \rho'^{(0)}_{X,V} - \vec{\nabla}_{v'} \rho'^{(0)}_{X,V} \right) = 0$$

$$\Big(\text{like} \quad \frac{d}{dx}(xf) - \frac{d^2 f}{dx^2} = 0$$

$$\frac{d}{dx}\left(xf - \frac{df}{dx} \right) = 0$$