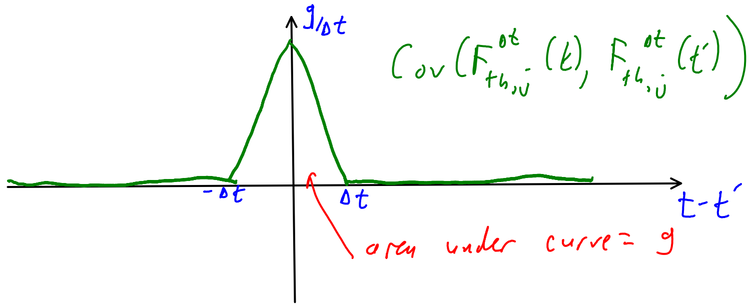
Working with Stochastic Differential Equations

Friday, February 20, 2009
12:00 PM

Homework 2 will be posted tonight, due Tuesday, March 3.

Continuing with our calculation of the covariance of the smoothed thermal force, by combining the various cases using the computation techniques from last time, we find:

$$\langle \vec{F}^{\Delta t}_{th}(t) \otimes \vec{F}^{\Delta t}_{th}(t') \rangle = \begin{cases} g\dfrac{(\Delta t - |t-t'|)}{(\Delta t)^2}\mathbb{I} & \text{for } |t-t'| < \Delta t \\ 0 & \text{else} \end{cases}$$



As $\Delta t \downarrow 0$, $\lim_{\Delta t \downarrow 0} \langle \vec{F}_{th}(t) \otimes \vec{F}_{th}(t') \rangle = g\mathbb{I}\,\delta(t-t')$

which agrees with what we wanted from the original physical equation.

So now making the replacement: $\vec{F}_{th}(t) = \sqrt{g}\,\dfrac{d\vec{W}}{dt}$

the Langevin equation becomes

$$\frac{d\vec{X}}{dt} = \vec{V}$$

$$m\frac{d\vec{V}}{dt} = -\gamma\vec{V} + \sqrt{g}\,\frac{d\vec{W}}{dt}$$

The usual practice in stochastic differential equations is to avoid discussion of derivatives of Wiener process because they don't exist in a usual sense -- formally multiply the differential equations by dt, which yields equations that can be more directly understood and numerically implemented in terms of increments of time and Wiener process:

$$d\vec{X} = \vec{V}\,dt$$

$$m\,d\vec{V} = -\gamma\vec{V}\,dt + \sqrt{g}\,d\vec{W}$$

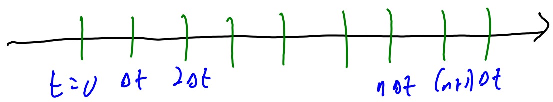
Langevin equation in SDE form

These equations can be given rigorous interpretations in terms of stochastic integrals (Gardiner Sec. 4.1), but it may be more intuitive to think of the simplest numerical discretization of this equation, which is actually a consistent approximation:

Euler-Marayama numerical scheme:

$$\vec{X}^{(n)} = \vec{X}(n\Delta t) \qquad \vec{V}^{(n)} = \vec{V}(n\Delta t)$$

$$\Delta\vec{W}^{(n)} = \vec{W}((n+1)\Delta t) - \vec{W}(n\Delta t)$$



$$\vec{X}^{(n+1)} - \vec{X}^{(n)} = \vec{V}^{(n)}\Delta t$$

$$m\left(\vec{V}^{(n+1)} - \vec{V}^{(n)}\right) = -\gamma\vec{V}^{(n)}\Delta t + \sqrt{g}\,\Delta\vec{W}^{(n)}$$

This gives a recursive rule for calculating the values of the variables X,V one time step after another. The only difference from the standard forward Euler method in ordinary differential equations is the need to add a random variable corresponding to the increments in the W iener process. We just use the rules for the Wiener process to deduce that:

$\{\Delta\vec{W}^{(n)}\}$ are independent Gaussian r.v.'s
with mean $\vec{0}$
covariance $\langle\Delta\vec{W}^{(n)} \otimes \Delta\vec{W}^{(n)}\rangle$
$= \Delta t\, \mathsf{I}$

In other words, each component of the vector-valued Wiener process increment is an independent Gaussian random variable with mean zero and variance Δt

We are of course here not considering issues of numerical stability and efficiency and accuracy which may make one choose a different numerical method for simulating stochastic differential equations, but at least the Euler-Marayama method is consistent in the sense that it is accurate for small enough time step. For more discussion on numerical methods for simulating stochastic differnetial equations, see
Kloeden and Platen, Numerical solution of Stochastic Differential Equations

How do we study stochastic differential equations analytically?

In many cases, one can use standard procedures for differential equations but the main difference in the calculus is that the increment dW is to be treated as a random increment with the following formal properties:

$d\vec{W}$ Gaussian r.v. with:

$$\langle d\vec{W}(t)\rangle = 0$$

$$\langle d\vec{W}(t) \otimes d\vec{W}(t')\rangle = \mathsf{I}\,\delta(t-t')\,dt\,dt'$$

These simple rules work in many calculations, including the present one. In more general systems, one has to be a bit careful.

To see a concise summary of the general rules for stochastic calculus, see Gardiner Secs. 4.1-4.3,'

The fundamental difficulty with stochastic differential equations arises when we have to deal with terms like

$$H(\{\vec{W}(t)\})\,d\vec{W}(t)$$

function or functional

For example, consider

$$dY = h(W(t))\, dW(t) \qquad Y(t) = Y(0) + \int_0 h(W(t'))\, dW(t')$$

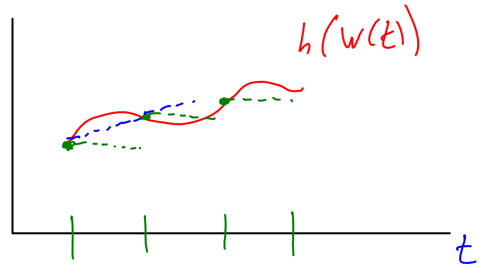
How do you discretize this numerically?

Ito: $$Y(t+\Delta t) - Y(t) \approx h(W(t))\,(W(t+\Delta t) - W(t)) \quad \sim \sqrt{\Delta t}$$

difference $\sim h'(W(t))\,\frac{(W(t+\Delta t) - W(t))}{2} \sim \sqrt{\Delta t}$

or

Stratonovich: $$Y(t+\Delta t) - Y(t) \approx \frac{h(W(t+\Delta t)) + h(W(t))}{2}\,(W(t+\Delta t) - W(t))$$



In ordinary differential equations ,these forward Euler and trapezoidal integration rules would converge to the same answer when the time step is made small, but this is not true for stochastic differential equations because the increment of the Wiener process only scales like the square root of the time increment.

This makes the two methods give a difference in the increment of Y which scales linearly with the time increment, and will accumulate to give an order one difference when these equations are integrated over a finite time interval with number of time steps proportional to $1/\Delta t$

There are two standard useful interpretations of stochastic differential equations, Ito and Stratonovich, and one must simply declare which one is using and then the stochastic calculus rules become well defined.

Which one should one use in a given model for physics, finance, biology, engineering, etc.?

Sometimes, one doesn't have to decide because both (all) interpretations are equivalent. This is the case for the Langevin equation.

But if it makes a difference, one needs to think about whether the dynamics of the system are such that the coefficient of the noise should be updated on the same time scale as the noise process. This is subtle, and depends on application.

Once one has made a choice of which interpretation is appropriate for a given model, then one can still choose in which framework to calculate. It turns out that the Ito framework is mathemtaically easier to work with, even though the calculus rules are a little different than normal. Stratonovich rules can be confusing even though they sometimes look easier.

The nice fact is that concrete rules exist for converting between Ito and Stratonovich forms of stochastic differential equations.

One good rule to know is when one doesn't worry about the interpretation -- when are they all equivalent?

This can be sorted out by just thinking about numerical discretization of the SDE and asking about the difference in the right hand side under various choices of numerical discretization. (Suffices usually just to think about forward Euler vs. trapezoidal rule).

If the different interpretations give rise to differences that are strictly higher order than linear in Δt then the interpretations will converge to the same answer for small time step. On the other hand, if the different interpretations create differences on the right hand side that can be as large as $O(\Delta t)$ then one has to be careful and specify the interpretation.

In particular, equations where the coefficient of the stochastic differential noise term does not depend on the noise or on the dependent variables, then all interpretations of the SDE are equivalent.

In our Langevin equation, for example, taking a trapezoidal rule rather than forward Euler rule would make change slike

$$\vec{V}^{(n)}\,\Delta t \to \frac{1}{2}\left(\vec{V}^{(n)} + \vec{V}^{(n+1)}\right)\Delta t$$

difference $\to 0$ as $\Delta t \to 0$

$\frac{\text{difference}}{\Delta t} \to 0$ as $\Delta t \to 0$

difference $\to 0$ as $\Delta t \to 0$

The Ito and Stratonovich interpretations of the Langevin equation are also the same even if $g = g(t)$:

$g_n = g(n\Delta t)$

$g_n \, \Delta W^{(n)}$ $\qquad$ $\frac{1}{2}(g_n + g_{n+1})\, \Delta W^{(n)}$

differ by $O(\Delta t)$

$$g_{n+1} = g((n+1)\Delta t) = g(n\Delta t) + g'(n\Delta t)\,\Delta t + \ldots$$
$$= g_n + g'(n\Delta t)\,\Delta t + \ldots$$
$$\frac{1}{2}(g_n + g_{n+1}) = g_n + g'(n\Delta t)\,\Delta t + \ldots$$

differ by $\sim O(\Delta t \sqrt{\Delta t}) \sim O((\Delta t)^{3/2})$

$\sim o(\Delta t)$

The difference in interpretations vanishes faster than linearly with the time step.

Our Langevin equation satisfies the property that the noise coefficient doesn't depend on the noise or the dependent variables so we can work with the simple stochastic calculus rules without worrying about interpretations.

$$m\, d\vec{V} = -\gamma \vec{V}\, dt + \sqrt{g}\, d\vec{W}(t)$$

We can try to work with this by treating the stochastic differential just as a prescribed external force even though it's random; it doesn't depend on the dependent variable. See Gardiner Sec. 4.4.4.

This looks like an inhomogenous linear differential equation which we can solve with integrating factors:

$$e^{+\frac{\gamma}{m}t}\, m\, d\vec{V} = -\gamma \vec{V} e^{+\frac{\gamma}{m}t}\, dt + \sqrt{g}\, e^{+\frac{\gamma}{m}t}\, d\vec{W}(t)$$

$$m\, d\left(\vec{V} e^{\frac{\gamma}{m}t}\right) = \sqrt{g}\, e^{\frac{\gamma}{m}t}\, d\vec{W}(t)$$

have to be a bit careful; make sure to include $dW\,dW$ contributions

$$d\vec{V} = -\frac{\gamma}{m}\vec{V}\, dt + \frac{1}{m}\sqrt{g}\, d\vec{W}$$
$$d\left(e^{\frac{\gamma}{m}t}\right) = \frac{\gamma}{m} e^{\frac{\gamma}{m}t}\, dt$$

To handle this, we generally speaking have to include products of differentials up to second order:

$$d\left(\vec{V} e^{\frac{\gamma}{m}t}\right) = d\vec{V}\, e^{\frac{\gamma}{m}t} + \vec{V}\, d\left(e^{\frac{\gamma}{m}t}\right) + d\vec{V}\, de^{\frac{\gamma}{m}t}$$

$\rightarrow = 0$ (negligible here)

just keep terms which are of the form $dW\, dW$

Here gives $dt\, dt$, $dt\, dW$ which can be neglected since $o(dt)$

$$d\left(\vec{V} e^{\frac{\gamma}{m}t}\right) = d\vec{V} e^{\frac{\gamma}{m}t} + \vec{V} d\left(e^{\frac{\gamma}{m}t}\right)$$

$$= \left(-\frac{\gamma}{m}\vec{V}\, dt + \frac{\sqrt{g}}{m} d\vec{W}\right) e^{\frac{\gamma}{m}t}$$

$$+ \vec{V}\left(\frac{\gamma}{m} e^{\frac{\gamma}{m}t}\, dt\right)$$

$$d\left(\vec{V} e^{\frac{\gamma}{m}t}\right) = \frac{\sqrt{g}}{m} e^{\frac{\gamma}{m}t}\, d\vec{W}(t)$$

perfect differentials

$$\vec{V}(t)\, e^{\frac{\gamma}{m}t} - \vec{V}(0)\, e^{0} = \frac{\sqrt{g}}{m}\int_0^t e^{\frac{\gamma}{m}t'}\, d\vec{W}(t')$$

$$\vec{V}(t) = \vec{V}(0)\, e^{-\frac{\gamma}{m}t} + \frac{\sqrt{g}}{m}\int_0^t e^{-\frac{\gamma}{m}(t-t')}\, d\vec{W}(t')$$

This derivation looked rather similar to the calculation one would do if the random force were deterministic prescribed function of time. Main subtlety is that in these stochastic differential equations, one must keep track of all terms that are as important as dt (but not higher order), and this means that one has to be alert for the presence of products of 2 dWs, in which case they must be incorporated as follows:

$$dW_i(t)\, dW_j(t) = \delta_{ij}\, dt$$

Gardiner Sec. 4.2

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if else} \end{cases}$$

Here we had to think about the product rule generalized to stochastic differentials, keeping track of second order terms in case any dW dW terms arise.

$$d\left(f(t)g(t)\right) = f(t+dt)\, g(t+dt) - f(t)\, g(t)$$

$$= \left(f(t) + df(t)\right)\left(g(t) + dg(t)\right) - f(t)\, g(t)$$

$$= f(t)\, dg(t) + df(t)\, g(t) + df\, dg$$

careful to include dW dW

include $d\tilde{W} dW$ terms.