TRUE/FALSE QUESTIONS

METRIC SPACES

(1) It is possible to define a metric on every non-empty set.  

(2) It is possible to define an infinite number of metrics on every set containing at least two elements.

(3) Let $(M, d_1)$ and $(M, d_2)$ denote metric spaces. Define $d : M \times M \to \mathbb{R}$ by $d(x, y) = d_1(x, y) + d_2(x, y)$, $\forall (x, y) \in M^2$. Then $(M, d)$ is a metric space.

(4) Let $(M, d_1)$ and $(M, d_2)$ denote metric spaces. Define $d : M \times M \to \mathbb{R}$ by $d(x, y) = d_1(x, y)d_2(x, y)$, $\forall (x, y) \in M^2$. Then $(M, d)$ is a metric space.

(5) Let $(M, d_1)$ denote a metric space and let $\lambda \in \mathbb{R}$. Define $d : M \times M \to \mathbb{R}$ by $d(x, y) = \lambda d_1(x, y)$, $\forall (x, y) \in M^2$. Then $(M, d)$ is a metric space.

(6) Let $(M, d)$ denote a metric space and let $x_0 \in M$. Define $d_1 : M \times M \to \mathbb{R}$ by $d_1(x, y) = d(x, x_0) + d(y, x_0)$, $\forall (x, y) \in M \times M$. Then $d_1$ is a metric on $M$.

(7) Let $(M, d_1)$ denote a bounded metric space such that $M$ contains at least two elements. Then there exists a metric space $(M, d)$ such that $10^6$ is the least upper bound of $\{d(x, y) : (x, y) \in M \times M\}$.

(8) Let $(M, d)$ denote a bounded metric space such that $M$ contains at least two elements. Let $10^6$ be the least upper bound of $\{d(x, y) : (x, y) \in M \times M\}$. Then there exists $(x_0, y_0) \in M \times M$ such that $d(x_0, y_0) = 10^6$.

(9) Let $(M, d)$ denote a metric space. If $d$ assumes an infinite number of values then $M$ is an infinite set.

(10) Let $(M, d)$ denote a metric space. If $M$ is an infinite set then $d$ assumes an infinite number of values.
(11) Let \((M, d)\) denote a metric space. Let \((x_0, y_0)\) and \((x_2, y_2)\) belong to \(M \times M\) such that \(d(x_0, y_0) < d(x_2, y_2)\). Then there exists \((x_1, y_1) \in M \times M\) such that \(d(x_0, y_0) < d(x_1, y_1) < d(x_2, y_2)\).

(12) Let \((M, d)\) denote a metric space. Let \((x_0, y_0)\) and \((x_2, y_2)\) belong to \(M \times M\) such that \(d(x_0, y_0) < d(x_2, y_2)\) and let \(\lambda \in \mathbb{R}\) such that \(d(x_0, y_0) < \lambda < d(x_2, y_2)\). Then there exists \((x_1, y_1) \in M \times M\) such that \(d(x_1, y_1) = \lambda\).

(13) A convergent sequence in a metric space is bounded.

(14) A convergent sequence in a metric space can have more than one limit.

(15) Every sequence in a bounded metric space is convergent.

(16) If every sequence in a metric space is bounded then the metric space is bounded.

(17) Every bounded sequence in a metric space is convergent.

(18) In a metric space, every subsequence of a convergent sequence is convergent.

(19) In a metric space, every Cauchy sequence is a convergent sequence.

(20) In a metric space an open ball is open.

(21) In a metric space a closed ball is closed.

(22) In a metric space the union of open sets is open.

(23) In a metric space the union of closed sets is closed.

(24) In a metric space the intersection of open sets is open.

(25) In a metric space the intersection of closed sets is closed.

(26) In a metric space there exist sets that are both open and closed.

(27) In a metric space the complement of an open set is open.
In a metric space the complement of a closed set is open.

In a metric space every non-empty subset has a boundary point.

In a metric space a closed set contains all of its boundary points.

In a metric space with an infinite number of elements there exists at least one limit point.

Let \( A \) and \( B \) denote subsets of a metric space and let \( x \) denote a boundary point of \( A \). Then \( x \) is a boundary point of \( A \cup B \).

In a metric space the set of boundary points of a set is a closed set.

In a metric space a set is closed if and only if its complement is open.

In a metric space a set consisting of a single point is a closed set.

In a metric space a set consisting of a single point is an open set.

In a metric space a set consisting of a finite number of points is a closed set.

In a metric space a set consisting of a countably infinite number of points is a closed set.

The norm on a normed vector space defines a metric on the vector space.

A convergent sequence in a metric space is a Cauchy sequence.

A sequence in a metric space contains an infinite number of distinct elements.

Let \( x \) denote a limit point of a set, \( S \), in a metric space. Then \( S \) contains an infinite sequence that converges to \( x \).

In a metric space the set of interior points of a set is an open subset of the metric space.

In a metric space every set has an interior point.
(45) In a metric space a set that has no interior points is closed.

(46) In a metric space the boundary of a set does not intersect its interior.

(47) Let \((M, d)\) denote a metric space and let \(f : M \to M\). Let \(U\) denote a closed subset of \(M\). If \(f\) is continuous then \(f^{-1}(U)\) is closed in \(M\).

(48) Let \((M, d)\) denote a metric space and let \(f : M \to M\). Let \(U\) denote a connected subset of \(M\). If \(f\) is continuous then \(f^{-1}(U)\) is a connected subset of \(M\).

(49) Let \((M, d)\) denote a metric space and let \(f : M \to M\). Let \(U\) denote a compact subset of \(M\). If \(f\) is continuous then \(f^{-1}(U)\) is a compact subset of \(M\).

(50) Let \((M, d)\) denote a metric space and let \(f : M \to M\). Let \(U\) denote an open subset of \(M\). If \(f\) is continuous then \(f(U)\) is an open subset of \(M\).

(51) Let \((M, d)\) denote a metric space and let \(f : M \to M\). Let \(U\) denote a closed subset of \(M\). If \(f\) is continuous then \(f(U)\) is a closed subset of \(M\).

(52) Let \(S\) denote a subset of \(\mathbb{R}\) defined by \(S = (0, 1]\). Then the set \((\frac{1}{2}, 1]\) is an open subset of the subspace \(S\).