

## TOTAL UNIMODULARITY

Theory for understanding why the solution to some linear programs are integer. Eg max flow, assignment problem.

Defn A square integer matrix  $B$  is called unimodular if  $\det(B)$  is  $\pm 1$ .

Defn An  $m \times n$  integral matrix  $A$  is totally unimodular (TU) if the determinant of each square submatrix of  $A$  is equal to  $0, 1$  or  $-1$ .

Note: if  $A$  is TU then every entry of  $A$  is either  $0, 1$  or  $-1$ .

If  $B$  is formed from a subset of  $m$  linearly indep cols of  $A$ , it determines the basic solution

$$x = B^{-1}b = \frac{B^{\text{adj}} b}{\det(B)}$$

$B^{\text{adj}}$  = adjoint of  $B$ .  
= transpose of matrix of cofactors.

Thus, if  $\det(B)$  is  $\pm 1$  then  $x$  is integer (prov  $b$  is integer)

S. we have

Thm If  $A$  is TU then all the vertices of  $\{x \in \mathbb{R}_+^n : Ax = b\}$  are integer for any integer vector  $b$ . ~~(not prov)~~

We also have:

Thm If  $A$  is TU then all the vertices of  $\{x \in \mathbb{R}_+^n : Ax \leq b\}$  are integer for any integer vector  $b$ . (No proof.)

Theorem The following statements are equivalent

- 1.  $A$  is TU
- 2.  $A^T$  is TU
- 3.  $[A, I]$  is TU
- 4. A matrix obtained by deleting a ~~unit~~ unit row (column) of  $A$  is TU
- 5. A matrix obtained by multiplying a row (column) of  $A$  by  $-1$  is TU
- 6. A matrix obtained by interchanging two rows (columns) of  $A$  is TU
- 7. A matrix obtained by duplicating columns (rows) of  $A$  is TU
- 8. A matrix obtained by a pivot operation on  $A$  is TU.

(Most of these follow straightforwardly from properties of determinants,

Thm An integer matrix  $A$  with  $a_{ij} = 0, 1, \text{ or } -1$  is TU if no more than two nonzero entries appear in any column, and if the rows of  $A$  can be partitioned into two sets  $I_1$  and  $I_2$  such that:

1. If a column has two entries of the same sign, their rows are in different sets
2. If a column has two entries of different signs, their rows are in the same set.

Proof By induction on size of submatrices.

$1 \times 1$  matrices:  $a_{ij} = 1, -1, 0$  so clear.

$k \times k$  matrices:

Let  $C$  be a  $k \times k$  submatrix of  $A$ .

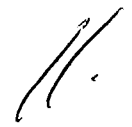
Case (i): If  $C$  has a column of all zeroes then it is singular, so  $\det(C) = 0$ .

Case (ii):  $C$  has a column with one nonzero entry:  
 Expand determinant along that column, result follows from induction hypothesis.

Case (iii): Each column of  $C$  has two nonzero entries.  
 Then

$$\sum_{i \in I_1} a_{ij} - \sum_{i \in I_2} a_{ij} = 0 \text{ for every column } j \text{ of } C.$$

ie a linear comb of rows is zero.  $\therefore \det(C) = 0.$



~~Lemma~~

Corollary Any LP of the form

$$\begin{array}{l} \min c^T x \\ Ax = b \\ x \geq 0 \end{array} \quad \text{or} \quad \begin{array}{l} \max c^T x \\ Ax \leq b \\ x \geq 0 \end{array}$$

where  $A$  is either

- 1) the node-arc incidence matrix of a directed graph, or
  - 2) the node-edge incidence matrix of an undirected bipartite graph,
- has only integer optimal ~~solutions~~ vertices.

This includes the LP formulations of:

shortest path, max-flow, assignment problem, weighted bipartite ~~max~~

(No proof.)

Theorem ~~If  $A$  is TU,~~

if  $P(b) = \{x \in \mathbb{R}^n : Ax \leq b\}$  is integral  $\forall b \in \mathbb{Z}^m$  for which it is not empty, then  $A$  is TU.

Not true for equality constraints  
 Interestingly theoretical question is whether a matrix  $A$  is TU iff  $\exists x \geq 0, Ax = b$  has only integral solutions.

eg:  $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$  (No proof.)

Theorem The following statements are equivalent:

- (i)  $A$  is TU
- (ii) For every  $J \subseteq N = \{1, \dots, n\}$ , there exist a partition  $J_1, J_2$  of  $J$  such that

$$\left| \sum_{j \in J_1} a_{ij} - \sum_{j \in J_2} a_{ij} \right| \leq 1 \text{ for } i=1, \dots, m.$$

Eg:  $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

$J = \{1, 2, 4\}$      $J_1 = \{1, 4\}$      $J_2 = \{2\}$ .

But  $J = \{1, 2\}$  : No partition.

Corollary

Interval matrices are TU

Matrix of 0's and 1's.

In each column, the 1's appear consecutively. Eg:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Proof

Put even rows in  $J_1$ , odd rows in  $J_2$ .  $\square$

Interval matrices used in scheduling: row  $i \leftrightarrow$  hour  $i$ . Workers work in shifts, so cover 8 (say) consecutive hours. Need each hour covered by a certain number of workers.