TOTAL UNIMODULARITY

Theory for understanding why the solution to some linear programs are integer. Eg max flow, assignment problem.

~~Defn A square integer matrix B is called unimodular if its determinant~~

Defn. An $m \times n$ integral matrix A is totally unimodular (TU) if the determinant of each square submatrix of A is equal to 0, 1 or -1.

Note: if A is TU then every entry of A is either 0, 1 or -1.

If B is formed from a subset of m linearly indep cols of A, it determines the basic solution

$$x = B^{-1} b = \frac{B^{adj} b}{\det(B)}$$

B^{adj} = adjoint of B.
= transpose of matrix of cofactors.

Thus, if $\det(B)$ is ± 1 then x is integer (prov b is integer)

So we have

Thm If A is TU then all the vertices of $\{x \in \mathbb{R}^n_+ : Ax = b\}$ are integer for any integer vector b. [illegible]

We also have:

Thm If A is TU then all the vertices of $\{x \in \mathbb{R}^n_+ : Ax \leq b\}$ are integer for any integer vector B. (No proof.)

<u>Theorem</u> The following statements are equivalent

1. A is TU
2. A^T is TU
3. $[A, I]$ is TU
4. A matrix obtained by deleting a ~~unit~~ $\overbrace{\text{unit row}}^{\text{row of } I}$ (column) of A is TU
5. A matrix obtained by multiplying a row (column) of A by -1 is TU
6. A matrix obtained by interchanging two rows (columns) of A is TU
7. A matrix obtained by duplicating columns (rows) of A is TU
8. A matrix obtained by a pivot operation on A is TU.

(Most of these follow straightforwardly from properties of determinants,

Thm An integer matrix A with $a_{ij} = 0, 1, \text{or} -1$ is TU if no more than two nonzero entries appear in any column, and if the rows of A can be partitioned into two sets I_1 and I_2 such that:

1. If a column has two entries of the same sign, their rows are in different sets
2. If a column has two entries of different signs, their rows are in the same set.

Proof By induction on size of submatrices.

1x1 matrices: $a_{ij} = 1, -1, 0$ so clear.

kxk matrices:

Let C be a kxk submatrix of A.

Case(i):
If C has a column of all zeroes then it is singular, so det(C) = 0

Case(ii):
C has a column with one nonzero entry:
Expand determinant along that column,
result follows from induction hypothesis.

Case(iii):
Each column of C has two nonzero entries.
Then

$$\sum_{i \in I_1} a_{ij} - \sum_{i \in I_2} a_{ij} = 0 \text{ for every column } j \text{ of } C.$$

ie a linear comb of rows is zero. ∴ det(C) = 0. //

Corollary Any LP of the form

$$\begin{array}{lcr} \min\ c^Tx & \quad\text{or}\quad & \max\ c^Tx \\ Ax = b & & Ax \le b \\ x \ge 0 & & x \ge 0 \end{array}$$

where A is either

1) the node-arc incidence matrix of a directed graph, or

2) the node-edge incidence matrix of an undirected bipartite graph,

has only integer optimal ~~solutions~~ vertices.

This includes the LP formulations of:

shortest path, max-flow, assignment problem, weighted bipartite ma

(No proof.)

Theorem ~~If A is TU,~~

If $P(b) = \{x \in \mathbb{R}^n_+ : Ax \le b\}$ is integral $\forall\, b \in \mathbb{Z}^m$ f
which it is not empty, then A is TU.

(No proof.)

Not true for equality constraints. E.g. $A = \left[\ [illegible]\ \right] \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ ([illegible])

Interesting theoretical question to determine when A is that $\{x \ge 0 : Ax = b\}$ has only integral BFS for all b.

Theorem The following statements are equivalent:

(i) A is TU

(ii) For every $J \subseteq N = \{1, \ldots, n\}$, there exists a partition J_1, J_2 of J such that

$$\left| \sum_{j \in J_1} a_{ij} - \sum_{j \in J_2} a_{ij} \right| \leq 1 \text{ for } i = 1, \ldots, m.$$

Eg: $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

$J = \{1, 2, 4\}$ $J_1 = \{1, 4\}$ $J_2 = \{2\}$.

But $J = \{1, 2\}$: No partition.

Corollary: Interval matrices are TU

Matrix of 0's and 1's.
In each column, the 1's appear consecutively Eg: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Proof: Put even rows in J_1, odd rows in J_2. $\blacksquare$

Interval matrices used in scheduling: row $i \leftrightarrow$ hour i. Workers work in shifts,
so cover 8 (say) consecutive hours.
Need each hour covered by a certain number of workers.