Polynomial Equivalence of Separation and Optimization.

The separation problem for a family of polyhedra.

An <u>instance</u> is given by an integer n, a description of a polyhedron $P \subseteq \mathbb{R}^n$ and a point x^* in $\mathbb{R}^n$.

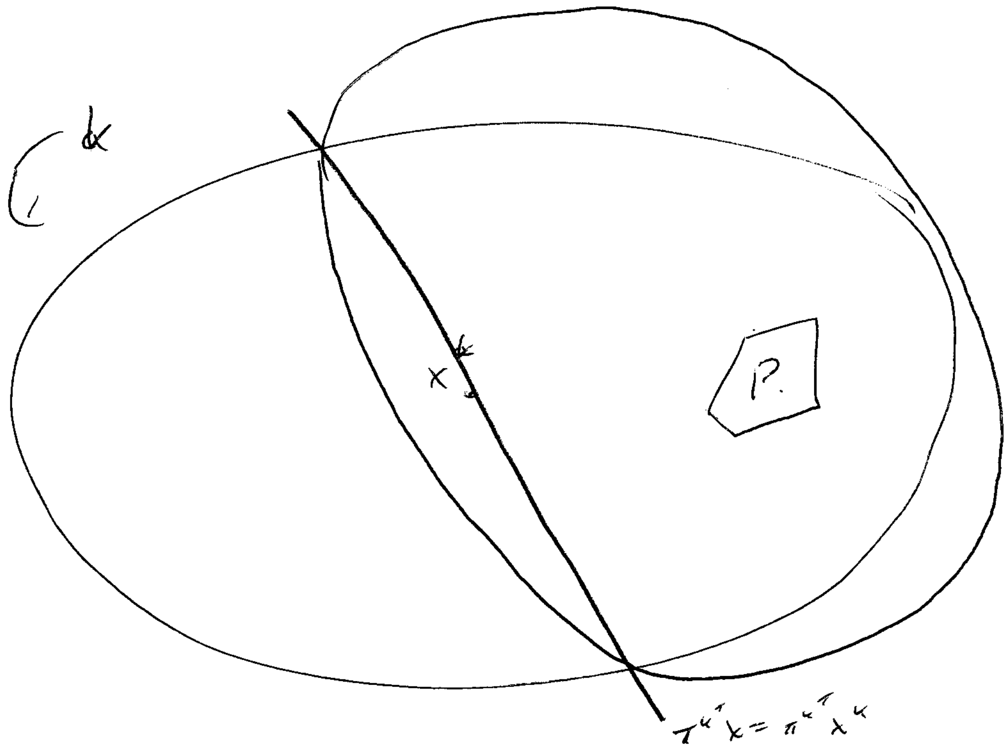
A <u>solution</u> is either the answer YES, x^* is in P

or <u>NO</u>, x^* is not in P, TOGETHER WITH a valid inequality $\pi^T x \leq \pi_0$ for P which is violated by x^*.

Eg: $P =$ convex hull of set of perfect matchings on graph with n edges.

We argue, ~~assuming the~~ that if the separation problem can be solved in polynomial time then the optimization problem $\max\{c^T x : x \text{ in } P\}$ can also be solved in polynomial time. We use the ellipsoid algorithm.

Note: ~~P is not~~ We do not know a description for P of the form $\{x \in \mathbb{R}^n : Ax \leq b\}$.



If x^k in P, cut on obj fn.

Else, can find a valid ineq for P in poly time, $\pi^{kT} x \le \pi^k_0$

~~So by arguments before, nu~~

Number of iters required of ellipsoid algorithm is polynomial in the original description of P. (really, in the dimension of the space, and not the number of constraints.)

~~Thus, total n~~

~~Number~~ of

Time required for an iteration is polynomial.

Thus, total time required is polynomial.

Note that we ~~may~~ get a poly time algo for the problem <u>even if</u> the number of facets of P is exponential.

Eg: Perfect matching problem:

~~max~~ min. $c^T x$
x is incidence vector of a perfect matching on $G=(V,E)$.

If x is a vector in $\mathbb{R}^{|E|}$, can solve the separation problem in $O(|V|^4)$ (Padberg & Rao).

Thus, by using ellipsoid algorithm, can solve the matching problem in time polynomial in $|V|$ (and the length of c).

But the polytope has exponentially many facets:
All odd set constraints

$$\binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{\lfloor \frac{n}{2} \rfloor (-1)}.$$

How can this happen?
Don't need all the facets for the optimal solution.
As a bare minimum, need no more than $|E|$.

Also have the other direction:

If the optimization problem can be solved in poly time, then so can the separation problem.

Node packing is NP-hard.

So probably can't solve the separation problem in polynomial time.

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p.37-38 Wolsey.

Wolsey defines two other useful properties (in addition to efficient optimization property and efficient separation property).

<u>Strong dual property</u>: For the given problem class, there exists a strong dual problem (D) $\min\{\omega(u) : u \in U\}$ allowing us to obtain optimality conditions that can be quickly verified:

$x^* \in X$ is optimal in P iff $\exists\, u^* \in U$ with $c^T x^* = \omega(u^*)$.

Note: (D) need not be an LP.

<u>Explicit Convex Hull Property</u>: A compact description of the convex hull $\operatorname{conv}(X)$ is known, which in principle allows us to replace every instance by the linear program $\max\{c^T x : x \in \operatorname{conv}(X)\}$.

If have explicit convex hull property, then it is likely the other properties will hold. Converse is not true.