

Primal-Dual Algorithms

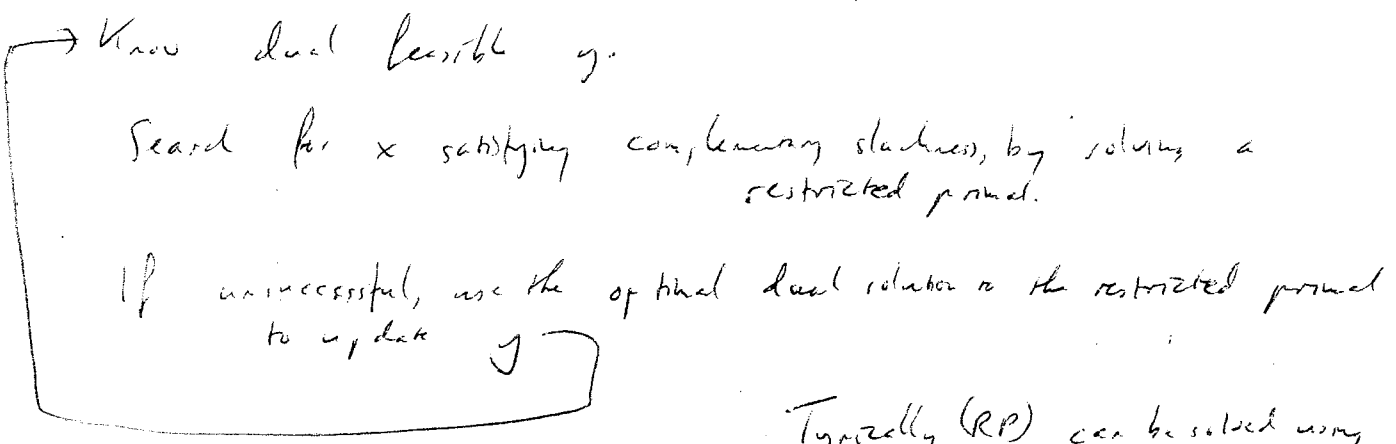
(P & S, Chapter 5.
Kochbaum, Chapter 4 by Goemans
& Williamson)

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned} \quad (P)$$

$$\begin{aligned} \max \quad & b^T y \\ \text{s.t.} \quad & A^T y \leq c \end{aligned}$$

Complementary slackness: $x_i (c_i - A_i^T y) = 0 \quad \forall i.$

Algorithm:



Typically (RP) can be solved using combinatorial techniques.

Eg. Assignment problem:

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & \sum_i x_{ij} = 1 \quad \forall j \\ & \sum_j x_{ij} = 1 \quad \forall i \\ & x_{ij} \geq 0 \end{aligned} \quad (P)$$

$$\begin{aligned} \max \quad & \sum u_j + \sum v_i \\ \text{s.t.} \quad & u_j + v_i \leq c_{ij} \end{aligned} \quad (D)$$

Assume $c_{ij} \geq 0$, to simplify the notation.

Initialize with $u_j = 0, \forall j, \quad v_i = 0 \quad \forall i$ (Better choices available)

Restricted primal:

$$\begin{aligned}
 \text{min} \quad & \sum s_j + \sum z_i \\
 \text{s.t.} \quad & \sum_i x_{ij} + s_j = 1 \quad \forall j \\
 & \sum_j x_{ij} + z_i = 1 \quad \forall i \\
 & x_{ij} = 0 \quad \text{if } u_j + v_i < c_{ij} \\
 & x_{ij} \geq 0, s_j \geq 0, z_i \geq 0
 \end{aligned} \tag{RP}$$

This is a maximum matching problem in a bipartite graph, so can be solved by combinatorial techniques. Optimal value is # unmatched vertices. If the optimal value is zero, we have primal feasibility, so we are done.

Otherwise, get optimal soln s_j^* , z_i^* .

Dual to (RP) is:

$$\begin{aligned}
 \text{max} \quad & \sum \bar{u}_j + \sum \bar{v}_i \\
 \text{s.t.} \quad & \bar{u}_j + \bar{v}_i \leq 0 \quad \text{if } u_j + v_i = c_{ij} \\
 & \bar{u}_j \leq 1 \\
 & \bar{v}_i \leq 1
 \end{aligned} \tag{RD}$$

(no constraint if $u_j + v_i < c_{ij}$, i.e. eliminate x_{ij} from (RP))

Can also be solved combinatorially.

Update: $(u, v) \leftarrow (u, v) + \alpha (\bar{u}, \bar{v})$ for some $\alpha > 0$, chosen using a minimum ratio test.

Can be used, e.g., for network design problems:
 Which edges do we build?

Constraints of the form

$$\sum_{e \in \delta(S)} x_e \geq f(S) \quad \text{for subsets } S \text{ of the vertices.}$$

E.g. each vertex must be connected to the rest of the graph,

$$\text{so } \sum_{e \in \delta(v)} x_e \geq 1 \quad \forall v \in V.$$

Integer programming formulation:

$$\min \sum c_e x_e$$

$$\text{s.t. } \sum_{e \in \delta(S)} x_e \geq f(S) \quad (P) \quad \text{for subsets } S \subseteq V$$

x_e binary.

Dual of LP relaxation (provided all $f(S) \leq 1$, so don't need $x_e \leq 1$ explicitly)

$$\max \sum f(S) y_S$$

$$\text{s.t. } \sum_{S: e \in \delta(S)} y_S \leq c_e \quad e \in E$$

$$y_S \geq 0.$$

Handwritten notes and scribbles:

- Point to vertices of C
- y_S when $S = \{v\}$
- (RP)
- $\min \sum c_e x_e + \sum y_S (\sum_{e \in \delta(S)} x_e - f(S))$
- $\text{s.t. } \sum_{e \in \delta(S)} x_e = f(S) \quad y_S \geq 0$
- $\sum_{e \in \delta(S)} x_e \geq f(S) \quad y_S = 0$
- $x_e \geq 0$

Goemans & Williamson:

Can use the primal-dual algorithm to get approximation schemes with provable bounds, for some choices of $f(S)$.

Primal dual method gives y_s .

→ We then want

$$\sum_{e \in \delta(S)} x_e = f(S) \quad \text{if } y_s > 0$$

Impose through objective \rightarrow $\sum_{e \in E} x_e = 0$ if $\sum_{S: e \in \delta(S)} y_s \leq c_e$

If these are not satisfied, find ^{minimal} a set $\{S\}$ that are violated.

Increase the corresponding y_s until one of the constraints

$$\sum_{S: e \in \delta(S)} y_s \leq c_e \quad \text{becomes active.}$$

Loop.

MATROIDS

Prototypes of independence systems with "nice" properties.

Defn: Let $N = \{1, \dots, n\}$ be a finite set, and let \mathcal{F} be a set of subsets of N . $\mathcal{I} = (N, \mathcal{F})$ is an INDEPENDENCE SYSTEM if $F_1 \in \mathcal{F}$, $F_2 \subseteq F_1 \Rightarrow F_2 \in \mathcal{F}$. Elements of \mathcal{F} are called INDEPENDENT SETS, and the remaining subsets of N are called DEPENDENT SETS.

Eg:

- (i) Acyclic subgraphs
- (ii) Linearly independent vectors (on a network)
- (iii) Matroids.

Def: Given an independence system $\mathcal{I} = (N, \mathcal{F})$, we say that F_i is a maximal independent set if $F_i \cup \{j\} \notin \mathcal{F}$ for any $j \in N \setminus F_i$.
 A maximal independent set T is maximal if $|S| \leq |T|$ for all $S \in \mathcal{F}$.

Def: $\mathcal{M} = (N, \mathcal{F})$ is a MATROID if \mathcal{M} is an independence system in which for any subset $T \subseteq N$, every independent set in T that is maximal in T has the same cardinality.

Eg: (i), (ii) ^{omitted in book} (i) is called GRAPHIC MATROID.

(ii) NO

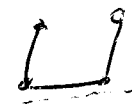
(iv) Uniform matroids: Every subset of size $\leq k$ is independent.

(v) Partition matroids: on disjoint base sets E_i .

$$E = \cup E_i$$

$F \subseteq E$ indep. if $|F \cap E_i| \leq 1$ for each i .

Eg: ~~graphical~~ ^{graphical} ~~single edge covering set~~

Eg: Need maximal = maximum "FOR EVERY SUBSET" because, eg
 matching on G vs matching on its subset 

Associate weights with elements of base set N

Theorem: Greedy algorithm solves the matroid problem

~~max $\{ \sum_{i=1}^k c_i x_i : x \text{ is incidence vector of independent set} \}$.~~

max $\{ \sum_{j \in S} c_j : S \text{ indep. set} \}$. Eg: spanning tree

Theorem: If (N, F) is an indep. sys. but not a matroid
 then there exists a weight function f which greedy fails.

Eg: matching

Theorem: Let $m(T)$ be the size of the largest independent set in $T \subseteq N$.

Then $m(T) + m(S) \geq m(T \cup S) + m(T \cap S)$. (~~Since~~ ^{Since} ~~from~~ ^{from} a function satisfying this condition is submodular.)

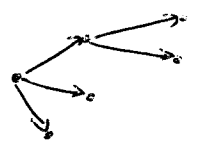
Eg: Spanning tree.

k-matroid intersection theorem

Given k matroids, $M_i = (N, F_i)$ $i=1, \dots, k$ on base set N, and weight vector $c_i, i \in N$, solve

$$\max_S \left\{ \sum_{j \in S} c_j : S \in \bigcap_{i=1}^k F_i \right\}$$

Application for $k=2$:
Find a branching matroid,
i.e., a spanning tree where at
most one edge enters each vertex



Solvable in poly time for $k=2$.

NP-hard for $k=3$.

Reduction from HAMILTONIAN PATH IN DIRECTED GRAPH.

Dual Matroids:

Call the independent sets of maximum cardinality the BASES OF A MATROID

Then all the complements of the bases form ~~the~~ the bases of another matroid: the dual matroid.

Eg: Uniform matroid: Dual matroid is also uniform.

• Partition matroid: Independent provided don't take all of a set C_i .

• Steiner subgraphs: Bases are complements of spanning trees

[Minimal dependent sets are circuits of a matroid.]

Dual ~~is~~ independent sets: sets of edges that do not disconnect the graph. (CUTSET MATROID)

Why does greedy work?

Assume there is a better soln.

There is something in this better soln that is not in the greedy soln.

Find the smallest dependent set when we add this extra element to the greedy soln.

Remove ~~the~~ worst element from the dependent set, giving a new indep set, better than greedy.

Greedy should have found this instead of the one it did.