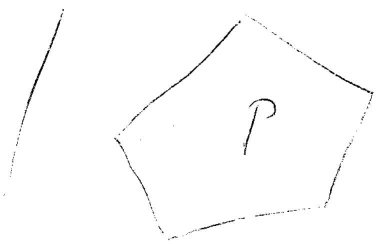
# Describing polyhedra by facets

Given a polyhedron $P = \{x \in \mathbb{R}^n : Ax \le b\}$

Which ineqs $a_i^T x \le b_i$ are necessary in description of $P$, which are redundant?




**Defn** The inequality $\pi^T x \le \pi_0$ (or $(\pi, \pi_0)$) is a valid inequality for $P$ if it is satisfied by all points in $P$.

**Defn** If $(\pi, \pi_0)$ is a valid inequality for $P$, and $F = \{x \in P : \pi^T x = \pi_0\}$ then $F$ is called a face of $P$, and we say that $(\pi, \pi_0)$ represents $F$. $F$ is proper if $F \neq \emptyset$ and $F \neq P$. If $F \neq \emptyset$, say $F$ supports $P$.

Note we can discard inequalities $a_i^T x \le b_i$ which are not support of $P$.

So from now on, assume all inequalities $a_i^T x \le b_i$ for $i \in M$ support $P$.

~~Prop If $P = \{x \in \mathbb{R}^n : Ax \le b\}$~~

**Prop** If $F$ is a nonempty face of $P$ then $F$ is a polyhedron and

$$F = \{x \in \mathbb{R}^n : a_i^T x = b_i \text{ for } i \in M_F^=,\ a_i^T x \le b_i \text{ for } i \in M_F^{\le}\},$$

where $M_F^= \supseteq M^=$ and $M_F^{\le} \subseteq M^{\le}$.

Number of distinct faces of $P$ is finite. **Proof** Exercise //.

<u>Defn</u> A face $F$ of $P$ is a <u>facet</u> of $P$ if $\dim(F) = \dim(P) - 1$.

<u>Prop</u> If $F$ is a facet of $P$, then exists some inequality $a_k^T x \le b_k$ for $k \in M^{\le}$ representing $F$.

<u>Proof</u> $\dim(F) = \dim(P) - 1$ ∴ $\operatorname{rank}(A_F^=, b_F^=) = \operatorname{rank}(A^=, b^=) + 1$.

∴ ~~...~~ we only need one inequality $a_k^T x \le b_k$ in $M^{\le}$ which is in $M_F^=$

This inequality represents $F$ //

<u>Prop</u> For each facet $F$ of $P$, one of the inequalities representing $F$ is necessary in the description of $P$.

<u>Proof</u> Exercise //.

<u>Prop</u> Every inequality $a_r^T x \le b_r$ for $r \in M^{\le}$ that represents a face of $P$ of dimension less than $\dim(P) - 1$ is irrelevant to the description of $P$.

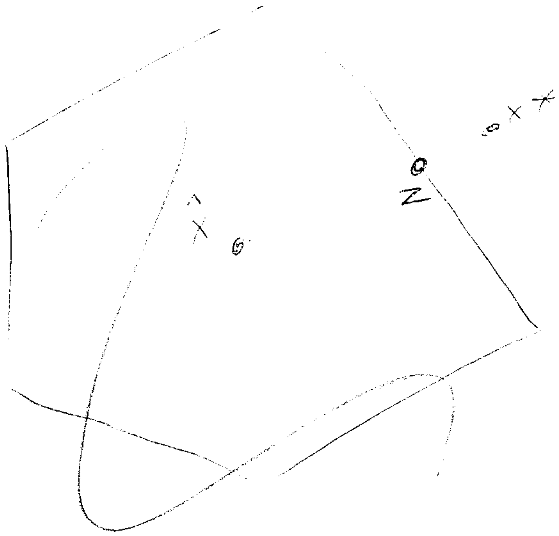
<u>Proof</u> Suppose $a_r^T x \le b_r$ represents face $F$ of $P$, $\dim(F) = \dim(P) - k$, $k > 1$. ~~...~~ inequality <u>not</u> irrelevant.

So $\exists\, x^* \in \mathbb{R}^n$ with $A^= x^* = b^=$, $a_i^T x^* \le b_i$ for $i \in M^{\le} \setminus \{r\}$,

and $a_r^T x^* > b_r$.

Let $\hat{x}$ be an inner point of $P$. Then on the line between $x^*$ and $\hat{x}$ there exists a point $z \in F$ satisfying $A^= z = b^=$, $a_i^T z < b_i$ for $i \in M^{\le} \setminus \{r\}$

and $a_r^T z = b_r$.

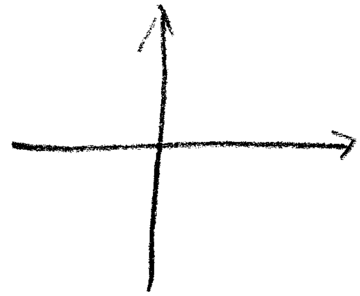



Hence, equality set of $F$ is $(A^=, b^=)$ and $(a_r, b_r)$, which is of rank $n - \dim(P) + 1$

Therefore, dimension of $F$ is $\dim(P) - 1$ ⨳. //

<u>Theorem</u> a) A full dimensional polyhedron $P$ has a unique (to within scalar multiplication) minimal representation by a finite set of linear inequalities. In particular, for each facet $F_i$ of $P$ there is an inequality $a_i^T x \le b_i$ (unique to within scalar multiplication) representing $F_i$ and $P = \{x \in \mathbb{R}^n : a_i^T x \le b_i \text{ for } i = 1, \dots, t\}$.

b) If $\dim(P) = n - k$ with $k > 0$, then $P = \{x \in \mathbb{R}^n : a_i^T x = b_i \text{ for } i = 1, \dots, k,\ a_i^T x \le b_i \text{ for } i = k+1, \dots, k+t\}$. For $i = 1, \dots, k$, $(a_i, b_i)$ are a maximal set of linearly independent rows of $(A^=, b^=)$, and for $i = k+1, \dots, k+t$, $(a_i, b_i)$ any inequality from the equivalence class of inequalities representing the facet $F_i$.



## Polarity

~~Char~~ Consider a polyhedron $\Pi$ whose ~~exten~~ feasible points are the valid ineq. of $P$. Will characterize facets of $P$ in terms of extreme ~~points~~ rays of $\Pi$.

**Def** $\Pi = \{(\pi, \pi_0) \in \mathbb{R}^{n+1} : \pi^T x - \pi_0 \leq 0 \text{ for all } x \in P\}$ is the polar of the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$

So $(\pi, \pi_0) \in \Pi$ iff $(\pi, \pi_0)$ is a valid ineq for $P$.

**Prop** Given a nonempty polyhedron $P \subseteq \mathbb{R}^n$ with $\text{rank}(A) = n$, $\Pi$ is a polyhedral cone described by

$$\pi^T x^k - \pi_0 \leq 0 \quad \forall k \in K$$

$$\pi^T \cancel{x} r^j \leq 0 \quad \forall j \in J$$

where $\{x^k\}_{k \in K}$, $\{r^j\}_{j \in J}$ are the extreme points and rays of $P$.

**Proof** Let $\Pi' = \{(\pi, \pi^0) \in \mathbb{R}^{n+1} : \pi^T x^k - \pi_0 \leq 0 \ \cancel{\forall x} \ \forall k \in K, \ \pi^T r^j \leq 0 \ \forall j \in J\}$

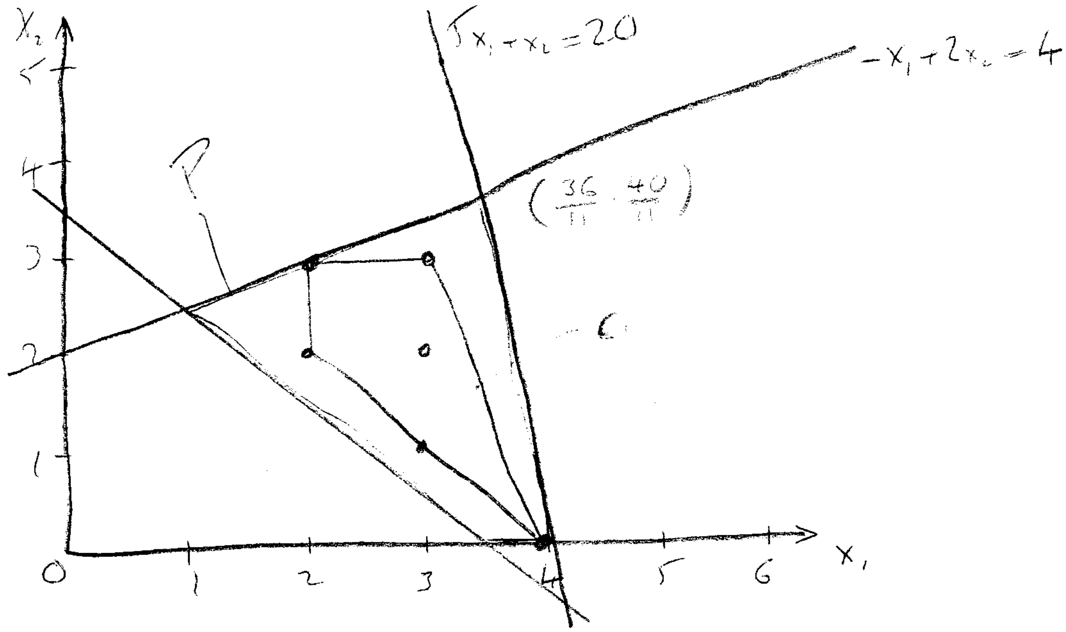
Show $\Pi' = \Pi$, by using Minkowski resolution thm //.

**Prop** If $\dim(P) = n$, $\text{rank}(A) = n$ and $\pi^* \neq 0$, then $(\pi^*, \pi_0^*)$ is an extreme ray of $\Pi$ if and only if $(\pi^*, \pi_0^*)$ defines a facet of $P$.

Example

Let $S$ be all points in $\mathbb{Z}^2_+$ satisfying:

$$\begin{aligned} -x_1 + 2x_2 &\leq 4 \\ 5x_1 + x_2 &\leq 20 \\ -2x_1 - 2x_2 &\leq -7 \end{aligned}$$




$$S = \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \end{pmatrix} \right\}$$

Note that $\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 4 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 4 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

So conv$(S)$ is a polytope defined by the four extreme points

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

conv(S) is given by

$$-x_1 \leq -2$$

$$x_2 \leq 3$$

$$-x_1 - x_2 \leq -4$$

$$3x_1 + x_2 \leq 12$$

Polar set $\Pi$ for $S$ (set of valid inequalities).

Can be found from extreme points of conv(S).

$\binom{2}{2}$: $2\pi_1 + 2\pi_2 - \pi_0 \leq 0$

$\binom{2}{3}$: $2\pi_1 + 3\pi_2 - \pi_0 \leq 0$

$\binom{3}{3}$: $3\pi_1 + 3\pi_2 - \pi_0 \leq 0$

$\binom{4}{0}$: $4\pi_1 - \pi_0 \leq 0$.

Consider the two valid inequalities:

$3x_1 + 4x_2 \leq 24$ ① (Goes through $\binom{8}{0}$ and $\binom{0}{6}$)

and $x_1 + x_2 \leq 6$ ② (Goes through $\binom{6}{0}$ and $\binom{0}{6}$)

$4x_1 + 4x_2 \leq 24$

Now $\{x : x_1 + x_2 \leq 6,\ x_1, x_2 \geq 0\} \subseteq \{x : 3x_1 + 4x_2 \leq 24,\ x_1, x_2 \geq 0\}$

So ② is "better" than ①.

Given $P = \{x \in \mathbb{R}^n_+ : Ax \leq b\}$, $S = P \cap \mathbb{Z}^n$.

Facets of conv(S) can be constructed iteratively using integrality and the linear inequality description of P.

So start with ineqs $Ax \leq b$

Progressively construct stronger ~~va~~ valid ineqs for conv(S) using integrality.

We know every point in P ~~set~~ (and also in conv(S)) satisfies

$$Ax \leq b$$
$$\cancel{x \geq 0}$$
$$-x \leq 0.$$

So any point in P also satisfies any linear combination of these inequalities. So get infinite family of valid ineqs:

$$\cancel{u^T A x \quad v^T}$$
$$u^T Ax - v^T x \leq u^T b + \alpha$$

for any nonnegative $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$.

<u>Propn</u> Let $\pi^T x \leq \pi_0$ be a valid inequality for $P = \{x \in \mathbb{R}^n_+ : Ax \leq b\}$. ~~It~~ If $P \neq \phi$, then $\pi^T x \leq \pi_0$ is either equivalent to or dominated by an inequality of the form $u^T Ax \leq u^T b$, $u \in \mathbb{R}^m_+$. //

We formalize the concept of one ~~ineq~~ valid inequality being better than another:

Consider two valid inequalities

$$\pi^T x \le \pi_0 \quad (A)$$
$$\gamma^T x \le \gamma_0 . \quad (B)$$

These ineqs are equivalent if $(\gamma, \gamma_0) = \mu(\pi, \pi_0)$ for some $\mu > 0$.

If they are not equivalent and there exists $\mu > 0$ such that $\gamma \ge \mu \pi$ and $\gamma_0 \le \mu \pi_0$, then

$$\{x \in \mathbb{R}^n_+ : \gamma^T x \le \gamma_0\} \subseteq \{x \in \mathbb{R}^n_+ : \pi^T x \le \pi_0\}$$

~~more restrictive on LHS less restrictive on~~

In this case, say ineq (B) dominates or is stronger than ineq (A), or that (A) is dominated by or is weaker than (B).

A maximal valid inequality is one that is not dominated by any other valid inequality.

In the example above, ineq (2) is a maximal ineq.

The ineq $x_1 \le 4$ defines a face, but is not maximal; it is dominated by $3x_1 + x_2 \le 12$.

Return to example on page 104.

$$P = \left\{ x \in \mathbb{R}^n_+ : \begin{array}{l} -x_1 + 2x_2 \le 4 \\ 5x_1 + x_2 \le 20 \\ -2x_1 - 2x_2 \le -7 \end{array} \right\}$$

a) Consider $u^T = \left(\frac{5}{11}, \frac{3}{22}, 0\right)$: (Bad case; doesn't give anything useful)

Get valid ineq (for $P$):

$$\left(-\frac{5}{11} + \frac{15}{22}\right)x_1 + \left(\frac{10}{11} + \frac{3}{22}\right)x_2 \le \frac{20}{11} + \frac{60}{22}$$

ie $$\frac{5}{22}x_1 + \frac{23}{22}x_2 \le \frac{100}{22}$$

Then round down LHS: $$x_2 \le \frac{100}{22}$$

Round down RHS $$x_2 \le 4$$ (~~Face of conv(S) but dominated by $5x_1 + x_2 \le 20$~~)

1 b) Consider $u^T = \left(\frac{4}{11} \;\; \frac{3}{11} \;\; 0\right)$ (Gives face of conv(S))

Get valid ineq (for $P$):

$$\left(-\frac{4}{11} + \frac{15}{11}\right)x_1 + \left(\frac{8}{11} + \frac{3}{11}\right)x_2 \le \frac{16}{11} + \frac{60}{11}$$

ie $$x_1 + x_2 \le \frac{76}{11}$$

Round down LHS: $$x_1 + x_2 \le \frac{76}{11}$$

Round down RHS: $$x_1 + x_2 \le 6$$

~~Facet~~ Face of conv(S)

2 c) Consider $u^T = (0 \quad 0 \quad \frac{1}{2})$ (Gives Facet)

Get valid ineq (for $P$):

$$-x_1 \quad - x_2 \quad \leq -\frac{7}{2}$$

Round down LHS: $-x_1 - x_2 \leq -\frac{7}{2}$

Round down RHS: $-x_1 - x_2 \leq -4$.

Facet of $\text{conv}(S)$.

Exercise: find vectors $u$ which give other facets of $\text{conv}(S)$.

d) $u^T = (\frac{1}{3} \quad \frac{2}{3} \quad 0)$ (Not useful)

Get valid ineq for $P$:

$$\frac{9}{3}x_1 + \frac{4}{3}x_2 \leq \frac{44}{3}$$

ie $3x_1 + x_2 \leq 14$ valid for $\text{conv}(S)$.

Worse than $5x_1 + x_2 \leq 20$, at least for $x$ in $P$.

4 (e) $u^T = (\frac{2}{7} \quad [illegible] \quad 0) \Rightarrow 3x_1 + x_2 \leq 13$ ([illegible] $\leq 12$) (cuts off ([illegible]))

3 (f) $u^T = ([illegible]) \Rightarrow x_1 \leq 3$ (facet)

5 (g) $u^T = ([illegible], 0, [illegible]) \Rightarrow -x_{[illegible]} \leq -1$ ([illegible] $\leq -2$)