Move these pages to just before page 90 — the end of polyhedral theory

Geometrical characterization of linear programming. (Intro to polyhedral theory)

Linear comb: A point $z \in \mathbb{R}^n$ is said to be a linear combination of the points x and y if $z = \lambda x + \mu y$, $\lambda, \mu \in \mathbb{R}$.

Defns: Subspace:

A set $S \subset \mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if every linear combination of points in S is also in S.

Linear indep: $\sum \lambda_i a^i = 0 \Rightarrow \lambda_1 = \dots = \lambda_k = 0$

Affine sets:

A point $z \in \mathbb{R}^n$ is an affine comb. of x and y if $z = \lambda x + (1-\lambda) y$ for some $\lambda \in \mathbb{R}$.

A set $M \subseteq \mathbb{R}^n$ is affine if every affine combination of points in M is also in M.

Affine indep: $a^1, \dots, a^k$ iff $a^2 - a^1, \dots, a^k - a^1$ lin indep

Half spaces:

$\{x : a^T x \geqslant \alpha\}$ is a half space

Hyperplane:

$\{x : a^T x = \alpha\}$ is a hyperplane. NB: this is an affine set.
$= \{x : a^T x \leqslant \alpha\} \cap \{x : a^T x \geqslant \alpha\}$

Polyhedron:

~~The~~ Finite intersection of half spaces.

Feasible region to an LP is a polyhedron.

Faces of a polyhedron: (Do *after* dimension).

Consider polyhedron P, hyperplane $H \equiv \{x : a^T x = \alpha\}$.

~~If $a^T x \geq \alpha \ \forall x \in P$~~

Let $Q = \cancel{\geq} P \cap H$. Q is a polyhedron.

If $a^T x \geq \alpha \ \forall x \in P$ (or $a^T x \leq \alpha \ \forall x \in P$) then Q is a *face* of P.

Dimension of a polyhedron:

Dimension of subspace = max no. of lin. indep vectors in it.

Every affine space is a translation of a subspace. (unique).

Dimension of affine space is dimension of the corresp. subspace. ~~(unique)~~

Dimension of a polyhedron is the dimension of the smallest affine space containing it. eg:

Then: Let P have dimension d.

Face of dim $d-1$ is a *facet*

Face of dim 1 is an *edge*

0 *vertex*. (or *extreme point*)

Thm Vertices $\Longleftrightarrow$ bfs.

Adjacent vertices $\Longleftrightarrow$ adjacent bfs.
(provided $B^{-1}b > 0$)

Polyhedral Theory and Cutting Plane Methods (NW I.4, II.1, II.2, I.4,

~~For IP, typically have a set of points~~

Would like to find a linear inequality description of the form $Ax \le b$ for an IP.

IP has, eg, set of feasi solns $S = \{x \in \mathbb{Z}^n_+ : Hx \le g\}$.

In order to get some understanding of what form this linear description would take, we look at polyhedral theory. Do pages 98–102 first.

Recall: Affine set M: $x, y \in M$, $\lambda \in \mathbb{R} \Rightarrow \lambda x + (1-\lambda) y \in M$

$y_1, \ldots, y_{k+1}$ aff indep $\Leftrightarrow$ $y_2 - y_1, \ldots, y_{k+1} - y_1$ lin indep.

Given ~~polyhedron P~~, subset $S \subseteq \mathbb{R}^n$,

dimension of S = dimension of smallest affine set containing it

Dimension of affine set = max # aff indep pts in the set $- 1$. Full-dimensional: dimension $= n$

Polyhedron: $P = \{x \in \mathbb{R}^n : Ax \le b\}$ (A, b) $m \times (n+1)$ matrix.

Replace by ...

Prop: If $\{x \in \mathbb{R}^n : Ax = b\} \neq \emptyset$, the maximum number of aff indep solns of $Ax = b$ is $n + 1 - \text{rank}(A)$.

Throughout, will assume P is rational ie it can be represented as above with (A, b) rational. Also assume P is stated as $\{x \in \mathbb{R}^n : Ax \le b\}$, (A, b) rational.

Let a_i^T be ith row of A.

Polyhedral ties between linear and integer programming

Let $S = \{x \in \mathbb{Z}^n : Ax \le b, x \geq 0\}$

Because of this, if P is nonempty it has extreme points.

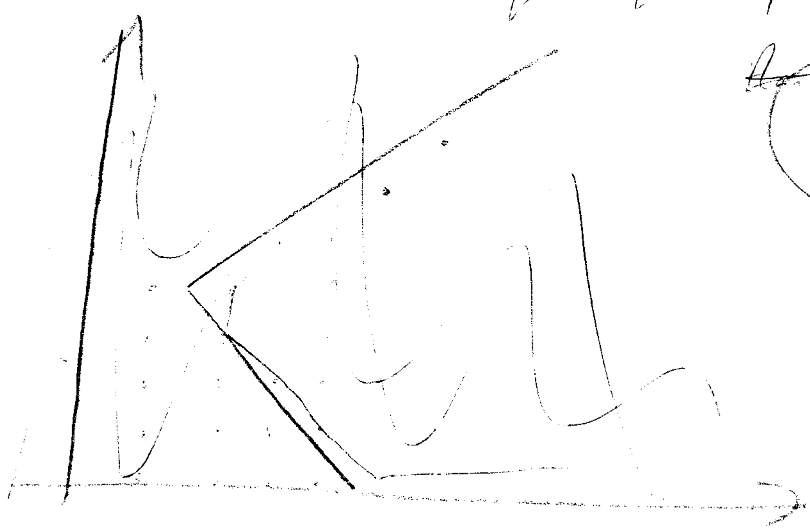
$= \mathbb{Z}^n \cap \{x \in \mathbb{R}^n_+ : Ax \le b\} = \mathbb{Z}^n \cap P$ P called polyhedron.

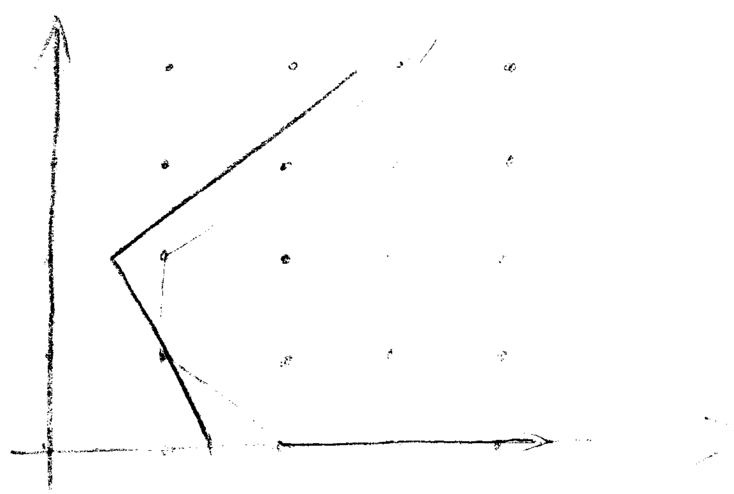
Theorem The convex hull of S is a polyhedron

Proof See N&W //   Obvious if S is finite

If P is unbounded, can show $\text{conv}(S)$ can be generated from a finite # points and rays.

~~Also~~ Extreme rays of $\text{conv}(S)$ are extreme rays of P.





Define:

(IP) $\max\{c^Tx : x \in S\}$ where $S = P \cap \mathbb{Z}^n$

$P = \{x : Ax \le b, x \geq 0\}$

(CIP) $\max\{c^Tx : x \in \text{conv}(S)\}$

<u>Theorem</u> Given $S = P \cap \mathbb{Z}^n \neq \emptyset$, $P = \{x \in \mathbb{R}^n_+ : Ax \le b\}$ and any c in $\mathbb{R}^n$, it follows that

(a) The objective value of IP is unbounded from above if and only if the objective value of CIP is unbounded from above.

(b) If CIP has a bounded optimal value, then it has an optimal solution (namely, an extreme point of conv(S)) that is an optimal solution to IP.

(c) If x° is an optimal solution to IP then x° is an optimal solution to CIP.

<u>Proof</u> Let z°, z^* be optimal values of IP, CIP respectively.

$S \subseteq \text{conv}(S) \Rightarrow z^\circ \leq z^*$ (*)

(a) (*) implies if $z^\circ = \infty$ then $z^* = \infty$.

Since r is extreme, it is rational. F

Conversely, if $z^* = \infty$ then there is an extreme ray r with $c^Tr > 0$ and x° ext. pt. of conv. Then $x^\circ + \theta r \in \text{conv}(S)$, and for appropriate choice of θ, $x^\circ + \theta r \in S$.

(b) Since conv(S) is polyhedron, if CIP has an optimal solution, it has an ext. optimal soln, say x°. Then $x^\circ \in S$, so $z^\circ \geq cx^\circ = z^*$. By (*), $z^\circ = z^*$

c) Follows from a) and b), since $x_0 \in conv(S)$. //

Corollary IP is either infeasible or unbounded or has an optimal solution.

Thus we can solve IP by solving the linear program CIP.

Problem: generally we do not know a set of linear inequalities that define $conv(S)$

So formulate IP by using some polyhedron $P = \{x \in \mathbb{R}^n : Ax \le b\}$ such that $S = P \cap \mathbb{Z}^n$. Reducing a IP to a LP corresponds to deducing ~~an appropriate set of constraints~~ a linear inequality representation of $conv(S)$. Note there are many possible polyhedra P such that $S = P \cap \mathbb{Z}^n$.

LP relaxation: IP is $\max\{c^T x : x \in P \cap \mathbb{Z}^n_{+}\}$. $P = \{x \in \mathbb{R}^n_+ : Ax \le b\}$

For the given P, Define the LP relaxation of IP:

$$\max \{c^T x : Ax \le b, x \in \mathbb{R}^n_+\} \qquad (LP)$$

Theorem Let z_{LP} = optimal value of LP

z_{IP} = — — IP.

a) If $P = \phi$ then $S = \phi$

b) If z_{LP} is finite then $S = \phi$ or z_{IP} is finite. If $S \neq \phi$ $z_{IP} \le z_{LP}$.

~~c) If $z_{LP} = \infty$, then $S = \phi$ or $z_{IP} = \infty$.~~

c) If x^0 is optimal for LP and $x^0 \in \mathbb{Z}_+^n$ then x^0 is optimal for IP.

Proof. a) $S \subseteq P$. $\therefore$ ~~&~~ $P = \phi \Rightarrow S = \phi$

b) If z_{LP} is finite:

$S \subseteq P$. $\therefore$ if $S \neq \phi$ then $z_{IP} \leq z_{LP}$.

c) $S \subseteq P$. & ~~x~~ x^0 is ~~better~~ at as good as any other point in P.

$\therefore$ x^0 is as good as any other point in S. //

Let $M = \{1, 2, \dots, m\}$

$M^= = \{ i \in M : a_i^T x = b_i \ \forall x \in P \}$ $\quad M^{\le} = M \setminus M^=$
$= \{ i \in M : a_i^T x < b_i \text{ for some } x \in P \}$.

Eg
$$\left.\begin{array}{r} x_1 + x_2 \le 1 \\ -x_1 - x_2 \le -1 \end{array}\right\} M^=$$
$$\left.\begin{array}{r} -x_1 \le 0 \\ -x_2 \le 0 \\ x_1 - x_2 \le 1 \end{array}\right\} M^{\le}$$

<u>Defn</u> $x \in P$ is an <u>inner point</u> of P if $a_i^T x < b_i \ \forall i \in M^{\le}$

$x \in P$ is an <u>interior point</u> of P if $a_i^T x < b_i \ \forall i \in M$.

Prop. Every nonempty polyhedron has an inner point

<u>Proof</u> Exercise // Take conv combs of pts which satisfy each constraint in $M^{\le}$ strictly.

Assume $P \ne \phi$. If $P \subseteq \mathbb{R}^n$ then $\dim(P) + \text{rank}(A^=, b^=) = n$.

<u>Proof</u> Since $P \ne \phi$, $\text{rank}(A^=) = \text{rank}(A^=, b^=) = n - k$, say.

Then $\exists$ $k+1$ aff indep solutions to $A^= x = 0$.

Let $y^1, \dots, y^{k+1}$ denote any such solutions, let $\hat{x}$ be an inner point of P.

Now for ε suff small, $\hat{x} + \varepsilon y^i$ are aff indep pts in P.

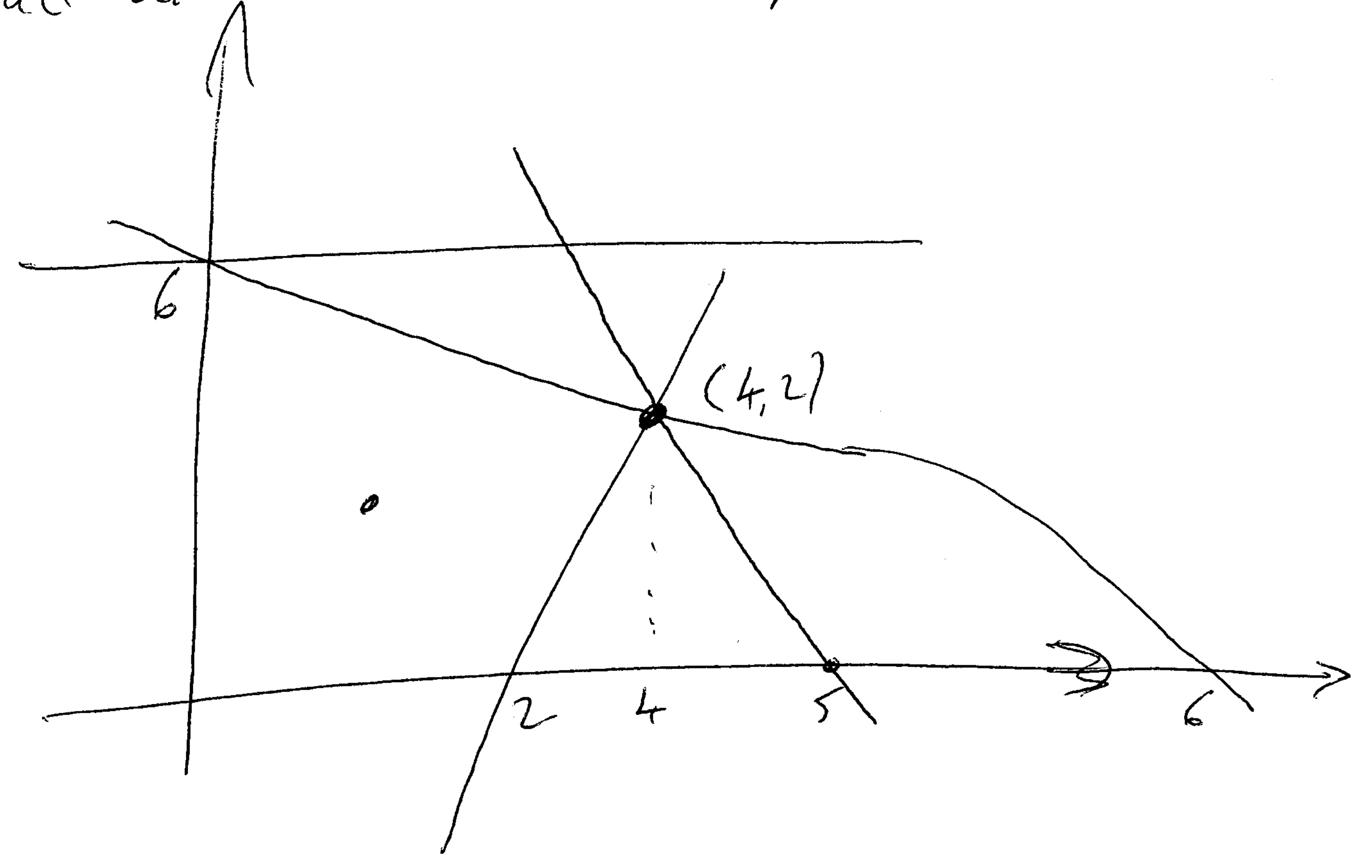
$\therefore \dim(P) \ge k$, so $\dim(P) + \text{rank}(A^=, b^=) \ge n$.

Now suppose $\dim(P) = k$ and $x^1, \dots, x^{k+1}$ are aff indep pts in P

Exercise follows from proposition //.

So if P is not full dimensional, at least one of ineqs $a_i^T x \le b_i$ is satisfied at equality by all $x \in P$.

Full dimensional example for page 91: JP 91a.



$$
\begin{aligned}
x_1 + x_2 &\le 6 \\
x_2 &\le 6 \\
x_1 - x_2 &\le 2 \\
2x_1 + x_2 &\le 10 \\
x_1 &\ge 0 \\
x_2 &\ge 0
\end{aligned}
$$

Interior point:

(1,3).

So rank $(A^=, b^=) = 0$.