Perfect Graphs

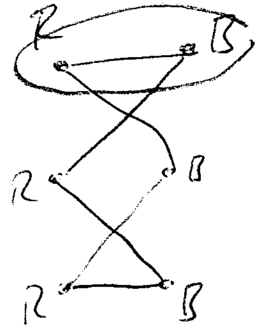
(Chudnovsky et al, Math Prog, 97(1-2), 2003, pp 405-422)

Defn A graph $G=(V,E)$ is PERFECT if $\chi(\bar{G})=\omega(\bar{G})$ for every induced subgraph $\bar{G}$ of G,

where $\chi(G)$ = CHROMATIC NUMBER of G
= least number of colors needed to color the vertices

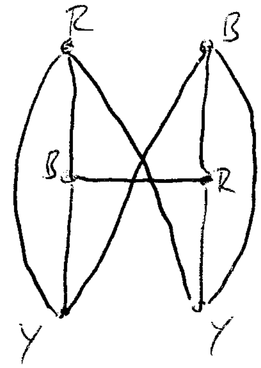
and $\omega(G)$ = the size of a maximum clique of G.

Eg: G =



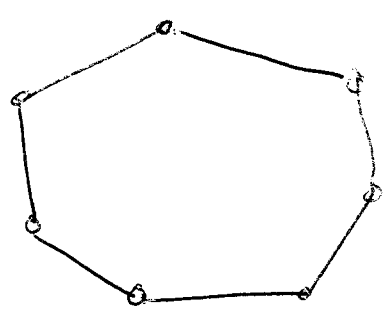
(bipartite graphs are perfect)

G =



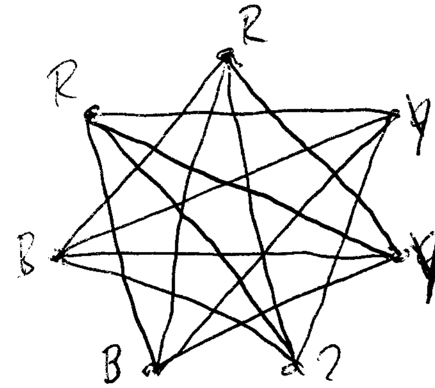
(complements of bipartite graphs are perfect)

Eg:



$\chi(G)=3$, $\omega(G)=2$

ODD-HOLES are not perfect



$\omega(G)=3$

ODD ANTIHOLES are not perfect.

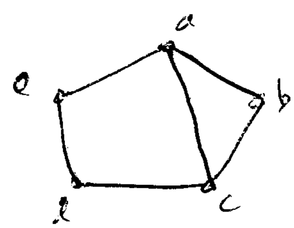
Packing LP:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq 1 \\ & x \geq 0 \end{aligned}$$

Every entry in A is 0 or 1.

Let rows of A correspond to incidence vectors of maximal cliques of a graph.

Eg:



$$A = \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

THM (Chvatal) The LP has an integral optimal solution for every objective c if and only if the underlying ~~the rows of A~~ graph is perfect.

Cliques in complement of G $\longleftrightarrow$ primal packing solutions

Colorings in complement of G $\longleftrightarrow$ dual solutions to packing LP.

THM A graph is perfect if and only if it has no odd hole and no odd antihole.

(Perfect graph conjecture of Berge (1961). Proved by Chudnovsky et al (2003).)