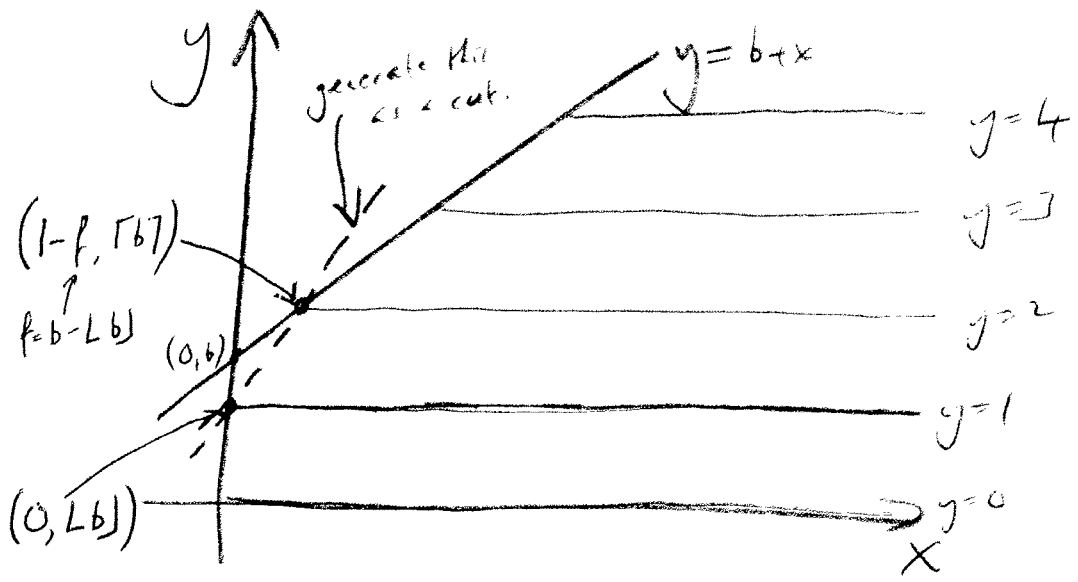


Cuts for Mixed Integer Programs (Wolsey, §8.7, §9.4)

Eg: $y \leq b + x$ where y integer, x continuous, both nonnegative.
 Can we strengthen this ~~cut~~ constraint?
 only really need $x \geq 0$



If $f = \lceil b - \lfloor b \rfloor \rceil =$ fractional part of b ,
 then $x = 1 - f$ and $y = b + x$ gives $y = b + 1 - f = b - f + 1 = \lfloor b \rfloor + 1 = \lceil b \rceil$.

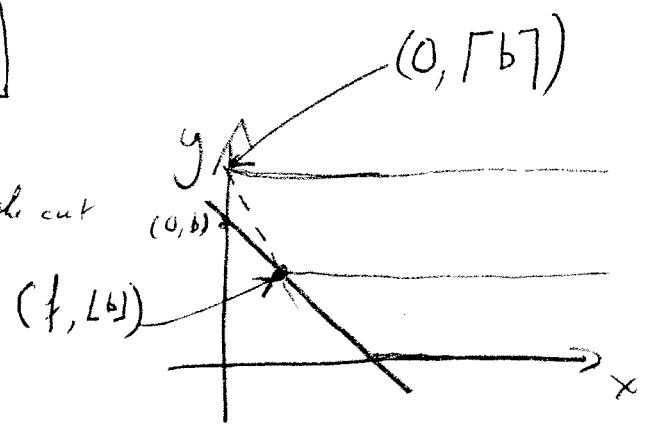
Thus, this additional constraint goes through
 $(0, \lfloor b \rfloor)$ and $(1-f, \lceil b \rceil)$.

So, intercept is $\lfloor b \rfloor$, slope is $\frac{1}{1-f}$, so given by

$$y \leq \lfloor b \rfloor + \frac{x}{1-f}$$

Similarly, $x + y \geq b$ leads to the cut

$$\frac{x}{f} + y \geq \lceil b \rceil$$



The Gomory Mixed Integer Cut:

Say we get a row in a tableau of the form:

$$(*) \quad y_{Bu} + \sum_{j \in N_1} \bar{a}_{uj} y_j + \sum_{j \in N_2} \bar{a}_{uj} x_j = \bar{a}_{u_0}, \quad (u \text{ ~~is~~ signifies the row of the tableau.)$$

where y_{Bu} is basic variable, integer

$y_j, j \in N_1$ are nonbasic integer variables,

$x_j, j \in N_2$ are nonbasic continuous variables

\bar{a}_{u_0} is fractional.

Let $f_j = \bar{a}_{uj} - \lfloor \bar{a}_{uj} \rfloor$ for $j \in N_1$.

$$f_0 = \bar{a}_{u_0} - \lfloor \bar{a}_{u_0} \rfloor$$

~~weaken (*) by rounding down~~

Divide N_1 into two sets: indices with $f_j > f_0$, indices with $f_j \leq f_0$.

Rewrite (*), and weaken it, as:

$$y_{Bu} + \sum_{\substack{j \in N_1 \\ f_j \leq f_0}} \lfloor \bar{a}_{uj} \rfloor y_j + \sum_{\substack{j \in N_1 \\ f_j > f_0}} \lceil \bar{a}_{uj} \rceil y_j$$

$$\leq \bar{a}_{u_0} - \sum_{j \in N_2} \bar{a}_{uj} x_j + \sum_{\substack{j \in N_1 \\ f_j > f_0}} (1 - f_j) y_j$$

Need this to ensure $x \geq 0 \rightarrow \bar{a}_{uj} < 0$

Now we appeal to the earlier result,

since the LHS must be integer, and the RHS is a constant + a continuous term:

$$y_{Ba} + \sum_{\substack{j \in N_1 \\ f_j \leq f_0}} \lfloor \bar{a}_{uj} \rfloor y_j + \sum_{\substack{j \in N_1 \\ f_j > f_0}} \lceil a_{uj} \rceil y_j \\ \leq \lfloor \bar{a}_{u_0} \rfloor - \sum_{\substack{j \in N_2 \\ \bar{a}_{uj} < 0}} \frac{\bar{a}_{uj}}{1-f_0} x_j + \sum_{\substack{j \in N_1 \\ f_j > f_0}} \frac{1-f_j}{1-f_0} y_j$$

Gather together terms, and place variables on LHS:

Notice that $\lceil a_{uj} \rceil - \frac{1-f_j}{1-f_0} = \lfloor a_{uj} \rfloor + 1 - \frac{1-f_j}{1-f_0}$ provided $f_j > 0$, which is implied by $f_j > f_0$.

$$= \lfloor a_{uj} \rfloor + \frac{1-f_0 - (1-f_j)}{1-f_0} = \lfloor a_{uj} \rfloor + \frac{f_j - f_0}{1-f_0}$$

Thus, we have:

$$y_{Ba} + \sum_{\substack{j \in N_1 \\ f_j \leq f_0}} \lfloor a_{uj} \rfloor y_j + \sum_{\substack{j \in N_1 \\ f_j > f_0}} \left(\lfloor a_{uj} \rfloor + \frac{f_j - f_0}{1-f_0} \right) y_j + \sum_{\substack{j \in N_2 \\ \bar{a}_{uj} < 0}} \frac{\bar{a}_{uj}}{1-f_0} x_j \leq \lfloor a_{u_0} \rfloor$$

this term makes the inequality stronger.

This cut does seem to be reasonably strong in practice

(see, e.g., Ceria et al MP 1998.)

Eg: ~~Row~~ Row of tableau:

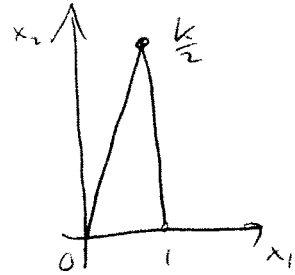
$$y_1 + \frac{1}{4} y_2 - \frac{1}{3} y_3 + \frac{5}{2} x_1 - \frac{3}{2} x_2 = \frac{3}{2} \quad (y_1 \text{ is basic var})$$

$$\text{Let: } y_1 + 0 y_2 + \left(-1 + \frac{\frac{2}{3} - \frac{1}{2}}{1 - \frac{1}{2}}\right) y_3 + \frac{\frac{5}{2}}{\frac{1-1/2}{1-1/2}} x_1 \leq 1$$

$$\text{or: } y_1 + \left(-1 + \frac{1/6}{1/2}\right) y_3 + \frac{5}{1-1/2} x_1 - 3 x_2 \leq 1$$

$$\text{or: } y_1 - \frac{2}{3} y_3 + \frac{5}{1-1/2} x_1 - 3 x_2 \leq 1$$

Eg:

$$\begin{array}{rcl} kx_1 + \boxed{x_2} + x_3 & = & k \\ -kx_1 + x_2 + x_4 & = & 0 \\ x_i \geq 0, \text{ integer.} \end{array} \quad \left. \vphantom{\begin{array}{rcl} kx_1 + \boxed{x_2} + x_3 & = & k \\ -kx_1 + x_2 + x_4 & = & 0 \\ x_i \geq 0, \text{ integer.} \end{array}} \right\} \begin{array}{l} \text{so } x_3 = k - kx_1 - x_2 \\ x_4 = kx_1 - x_2 \end{array}$$


$$\begin{array}{rcl} kx_1 + x_2 + x_3 & = & k \\ \boxed{-2kx_1} - x_3 + x_4 & = & -k \end{array}$$

$$\begin{array}{rcl} * & x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 & = \frac{k}{2} \quad (1) \\ x_1 + \frac{1}{2k}x_3 - \frac{1}{2k}x_4 & = & \frac{1}{2} \quad (2) \end{array}$$

$x_i \geq 0$

Gomory cut from (2):

$$\frac{1}{2k}x_3 + \frac{2k-1}{2k}x_4 \geq \frac{1}{2} \quad (3) \Rightarrow x_3 + (2k-1)x_4 \geq k$$

Substituting for x_3, x_4 :

$$k - kx_1 - x_2 + (2k-1)kx_1 - (2k-1)x_2 \geq k$$

$$\Rightarrow \cancel{2k^2x_1 - 2kx_1} - 2kx_2 \geq 0$$

$$\Rightarrow (k-1)x_1 - x_2 \geq 0, \text{ as we found earlier.}$$

If we strong duality from (2):

$$x_1 + \cancel{\frac{1}{2k} A_3} + \left(-1 + \frac{\frac{2k-1}{2k} - \frac{1}{2}}{\frac{1}{2}}\right) x_4 \leq 0$$

$$\Rightarrow x_1 + \cancel{\frac{1}{2k} A_3} + \left(-1 + \frac{k-1}{k}\right) x_4 \leq 0$$

$$\Rightarrow x_1 + \cancel{\frac{1}{2k} A_3} - \frac{1}{k} x_4 \leq 0$$

Substituting for x_3, x_4 :

$$x_1 + \cancel{\frac{1}{2k} (kx_1 - x_2)} - \frac{1}{k} (kx_1 - x_2) \leq 0$$

$$\Rightarrow x_1 + \cancel{\frac{1}{2k} kx_1} - \frac{1}{k} kx_1 + \frac{x_2}{k} \leq 0$$

$$\cancel{\frac{x_1}{2}} - \frac{x_1}{k} + \frac{x_2}{k}$$

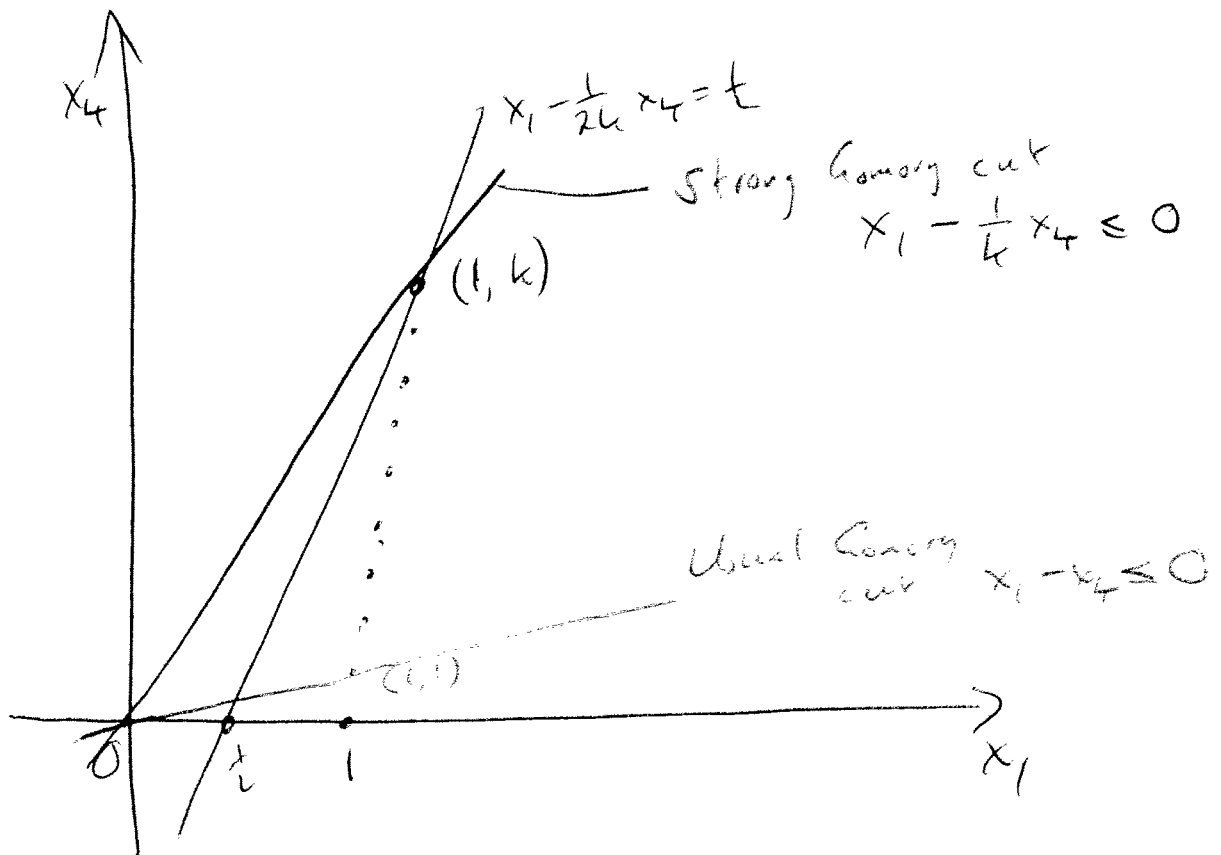
$$\cancel{\frac{x_1}{2}} - \frac{x_1}{k} + \frac{x_2}{k}$$

\Rightarrow

$$\boxed{x_2 \leq 0}$$

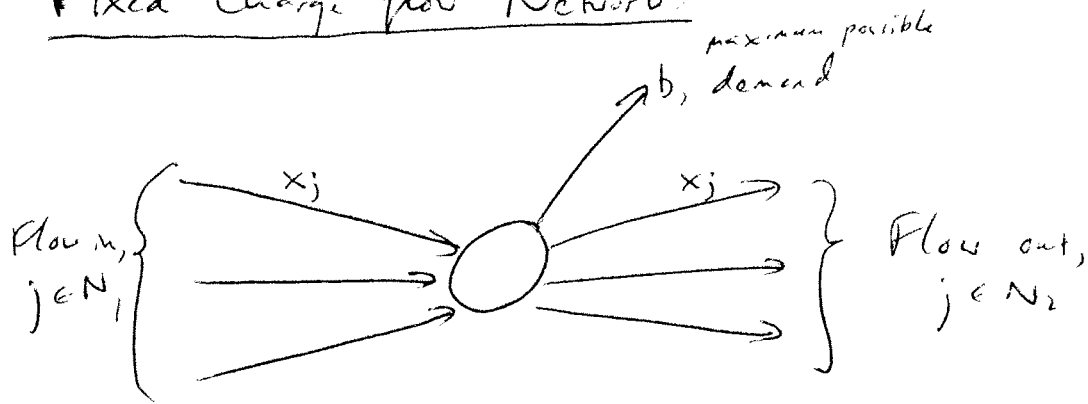
Graphically:

(2) can be weakened to $x_1 - \frac{1}{2k} x_4 \leq \frac{1}{2}$



Any C-G cut takes a valid constraint (eg, $x_1 - x_4 \leq \frac{1}{2}$) and pushes it until it hits an integer point (eg $x_1 = 1, x_4 = 1$), even if that point violates the original constraint.

The Strong Gomory Cut ignores the integrality of x_4 , anchors the current cut where it crosses $x_1 = 1$, and pushes until it hits an integer point with $x_1 = 0$.

Fixed Charge Flow Networks:

Flow conservation:

$$\sum_{j \in N_1} x_j \leq b + \sum_{j \in N_2} x_j$$

Pay a cost c_j for using an arc, and the arc has capacity a_j .

So use 0-1 variable y_j : $x_j \leq a_j y_j$

place term $c_j y_j$ in objective function.

b and a_j may be fractional, so x may also be fractional.

Flow cover inequalities:

Defn A set $C = C_1 \cup C_2$ with $C_1 \subseteq N_1$, $C_2 \subseteq N_2$ is a GENERALIZED

cover for x if $\sum_{j \in C_1} a_j - \sum_{j \in C_2} a_j = b + \lambda$ for some $\lambda > 0$.

Notice that we can't have $x_j \geq a_j \forall j \in C_1 \cup C_2$
 $= \begin{cases} a_j & \forall j \in C_1 \cup C_2 \\ 0 & \text{otherwise} \end{cases}$

If we have a cover, the inequality

$$\sum_{j \in C_1} x_j + \sum_{j \in C_2} (a_j - \lambda)^+ (1 - y_j) \leq b + \sum_{j \in C_2} a_j + \lambda \sum_{j \in L_2} y_j + \sum_{j \in N_1 \setminus (C_1 \cup C_2)} x_j$$

is valid, for any $L_2 \subseteq N_2 \setminus C_2$. (Wolsey, Prop 9.6)
 (Cappato, 4.1.1, 4.1.2)

Separation routines to find such ineqs, which are violated are described in Wolsey,

Eg: $x_1 + x_2 + x_3 \leq 3 + x_4 + x_5$

$x_1 \leq 2y_1, x_2 \leq y_2, x_3 \leq 6y_3, x_4 \leq 3y_4, x_5 \leq 5y_5$

① $C_1 = \{1, 3\}, C_2 = \{4\}, \lambda = 2.$

Get

$x_1 + x_3 + (2-2)^+(1-y_1) + (6-2)^+(1-y_3)$

$\leq 3 + 3 + \underbrace{2(y_4)}_{2y_5}$

if $L_2 = \{5\}$.
 Could replace this with
 ~~$2y_5 + x_4 + (6-2)^+(1-y_3)$~~
 ~~$2y_5 + y_4 + (6-2)^+(1-y_3)$~~
 ~~$x_4 + x_5$~~ (c.c. = 9)

② i.e., $x_1 + x_3 + 0(1-y_1) + 4(1-y_3) \leq 6 + 2y_5$

i.e., $x_1 + x_3 - 4y_3 - 2y_5 \leq 2$ (1)

Seems reasonable, since $x_1 = 3, x_3 = 6$ (i.e., at their limits),

then forces $y_5 = 1$. (rather than $y_5 = \frac{2}{5}$ with $x_5 = 2$)

Note that this is infeasible, as shown by the version with x_5 :

$$\begin{matrix} x_1 & + & x_3 & - & y_1 & - & 4y_3 & \leq & 1 & + & x_5 & (2) \\ \hline 1 & & 1 & & 1 & & 1 & & & & 1 & \\ \hline 2 & & 6 & & 1 & & 1 & & & & 1 & \end{matrix}$$
 With $x_1 = 3, x_3 = 6$, have lhs ≥ 3
 But rhs ≤ 2 .

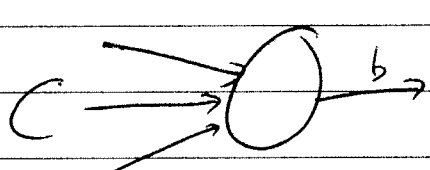
If change bound on x_5 to $x_5 \leq 3y_5$, (1) & (2) are not changed, but we can now satisfy (2).

The generalised flow cover inequality is equation (4.5) in section II.6.4 on page 499 of Nemhauser & Wolsey. Its proof is Exercise 32 on page 531.

A simplified version is proved in Prop 4.3 in section II.2.4 on page 283.

An even simpler version is Prop 4.2 on page 282, § II.2.4.

This version is:



The diagram shows a central circle representing a node. Three arrows point towards it from the left, with the label 'C' to their left. One arrow points away from the node to the right, labeled 'b'.

$$\sum_{j \in C} x_j \leq b - \sum_{j \in C} (a_j - \lambda)^+(1 - y_j)$$

where $\lambda = \sum_{j \in C} a_j - b > 0$.

Clearly valid if all $y_j = 1$.

If exactly one y_j is 0, and correspondingly $a_j > \lambda$,

then max flow is $\sum_{j \in C \setminus j} a_j = b + \lambda - a_j$ by defn of λ .

If at least two y_j 's are 0, max flow is further reduced by another a_j , so reducing it by $(a_j - \lambda)^+$ is conservative.