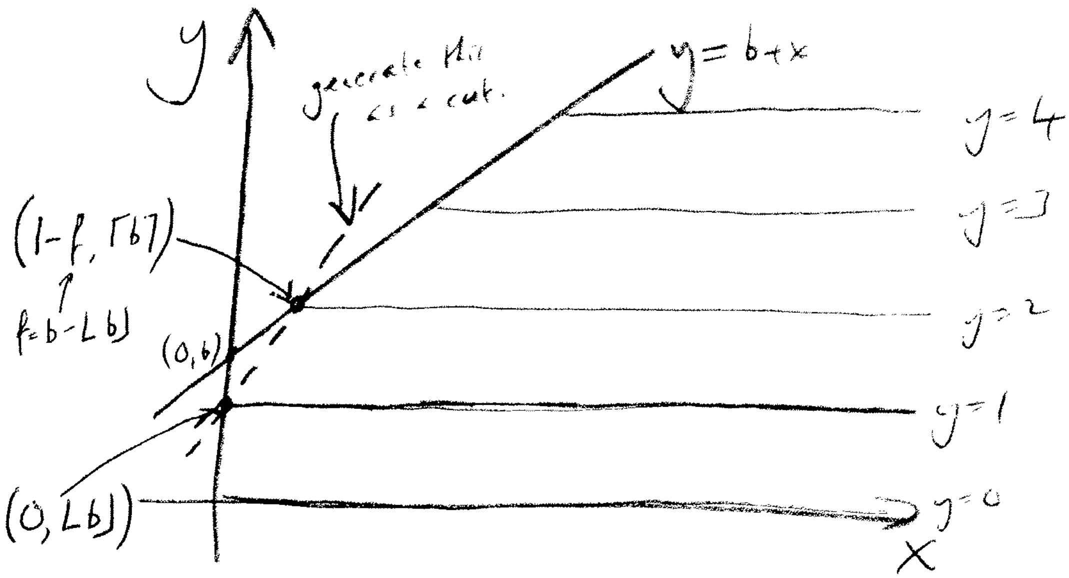
Cuts for Mixed Integer Programs (Wolsey, §8.7, §9.4)

Eg: $y \le b + x$ where y integer, x continuous, both nonnegative.

only really need $x \ge 0$

Can we strengthen this ~~cut~~ constraint?



If $f = \cancel{fra}\, b - \lfloor b \rfloor$ = fractional part of b,

then $x = 1 - f$ and $y = b + x$ gives $y = b + 1 - f = b - f + 1 = \lfloor b \rfloor + 1 = \lceil b \rceil$

Thus, this additional constraint goes through

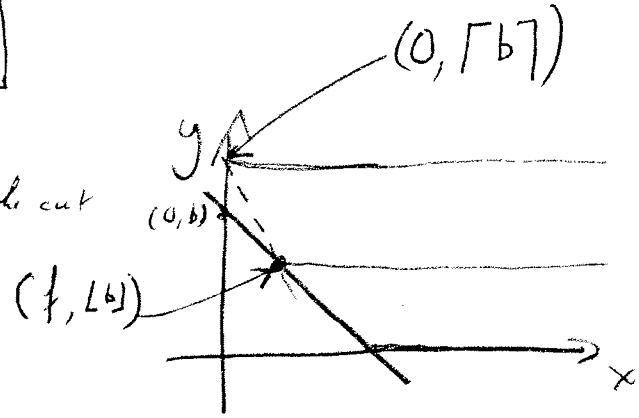
$(0, \lfloor b \rfloor)$ and $(1-f, \lceil b \rceil)$.

So, intercept is $\lfloor b \rfloor$, slope is $\frac{1}{1-f}$, so given by

$$\boxed{y \le \lfloor b \rfloor + \frac{x}{1-f}}$$

Similarly, $x + y \ge b$ leads to the cut

$$\frac{x}{f} + y \ge \lceil b \rceil$$



The Gomory Mixed Integer Cut:

Say we get a row in a tableau of the form:

(*) $$y_{B_u} + \sum_{j\in N_1} \bar{a}_{uj}\, y_j + \sum_{j\in N_2} \bar{a}_{uj}\, x_j = \bar{a}_{u0},$$ (u ~~sig~~ signifies the row of the tableau.)

where y_{B_u} is basic variable, integer

$y_j, j\in N_1$ are nonbasic integer variables,

$x_j, j\in N_2$ are nonbasic continuous variables

$\bar{a}_{u0}$ is fractional.

Let $f_j = \bar{a}_{uj} - \lfloor \bar{a}_{uj} \rfloor$ for $j\in N_1$.

$f_0 = \bar{a}_{u0} - \lfloor \bar{a}_{u0} \rfloor$

~~Weaken (*) by rounding down~~

Divide N_1 into two sets: indices with $f_j > f_0$, indices with $f_j \le f_0$.

Rewrite (*), and weaken it, as:

↓ because round down here

$$y_{B_u} + \sum_{\substack{j\in N_1\\ f_j\le f_0}} \lfloor \bar{a}_{uj} \rfloor y_j + \sum_{\substack{j\in N_1\\ f_j > f_0}} \lceil \bar{a}_{uj} \rceil y_j$$

$$\le \bar{a}_{u0} - \sum_{j\in N_2} \bar{a}_{uj}\, x_j + \sum_{\substack{j\in N_1\\ f_j > f_0}} (1-f_j)\, y_j$$

Need this to ensure 'x' ≥ 0 $\longrightarrow$ $\bar{a}_{uj} < 0$

Now we appeal to the earlier result,

since the LHS must be integer, and the RHS is a constant + a continuous term:

$$y_{B_u} + \sum_{\substack{j \in N_1 \\ f_j \leq f_0}} \lfloor \bar{a}_{uj} \rfloor y_j + \sum_{\substack{j \in N_1 \\ f_j > f_0}} \lceil a_{uj} \rceil y_j$$

$$\leq \lfloor \bar{a}_{u0} \rfloor - \sum_{\substack{j \in N_2 \\ \bar{a}_{uj} < 0}} \frac{\bar{a}_{uj}}{1-f_0} x_j + \sum_{\substack{j \in N_1 \\ f_j > f_0}} \frac{1-f_j}{1-f_0} y_j$$

Gather together terms, and place variables on LHS:

Notice that $\lceil a_{uj} \rceil - \frac{1-f_j}{1-f_0} = \lfloor a_{uj} \rfloor + 1 - \frac{1-f_j}{1-f_0}$ provided $f_j > 0$, which is implied by $f_j > f_0$.

$$= \lfloor a_{uj} \rfloor + \frac{1-f_0-(1-f_j)}{1-f_0} = \lfloor a_{uj} \rfloor + \frac{f_j - f_0}{1-f_0}$$

Thus, we have:

$$y_{B_u} + \sum_{\substack{j \in N_1 \\ f_j \leq f_0}} \lfloor a_{uj} \rfloor y_j + \sum_{\substack{j \in N_1 \\ f_j > f_0}} \Big(\lfloor a_{uj} \rfloor + \underbrace{\frac{f_j - f_0}{1-f_0}}_{\text{this term makes the cut a little stronger.}} \Big) y_j + \sum_{\substack{j \in N_2 \\ a_{uj} < 0}} \frac{\bar{a}_{uj}}{1-f_0} x_j \leq \lfloor a_{u0} \rfloor$$

This cut does seem to be reasonably strong in practice (see, eg, Cena et al MP 1998.)

Eg: ~~Get~~ Have row of tableau:

$$y_1 + \frac{1}{4} y_2 - \frac{1}{3} y_3 + \frac{5}{2} x_1 - \frac{3}{2} x_2 = \frac{3}{2} \qquad (y_1 \text{ is basic var})$$

Get: $y_1 + 0 y_2 + \left(-1 + \frac{\frac{2}{3} - \frac{1}{2}}{1 - \frac{1}{2}}\right) y_3 + \cancel{\frac{\frac{5}{2}}{1-\frac{1}{2}} x_1} - \frac{3/2}{1-\frac{1}{2}} x_2 \leq 1$

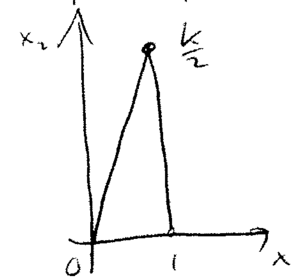
or: $y_1 + \left(-1 + \frac{1/6}{1/2}\right) y_3 + \cancel{5x_1} - 3x_2 \leq 1$

or: $y_1 - \frac{2}{3} y_3 + \cancel{5x_1} - 3x_2 \leq 1$

Eg:

$$\begin{aligned} kx_1 + x_2 + x_3 &= k \\ -kx_1 + x_2 + x_4 &= 0 \end{aligned} \quad \Bigg\} \quad \text{so } \begin{aligned} x_3 &= k - kx_1 - x_2 \\ x_4 &= kx_1 - x_2 \end{aligned}$$

$x_i \geq 0$, integer.



$$\begin{aligned} kx_1 + x_2 + x_3 &= k \\ -2kx_1 - x_3 + x_4 &= -k \end{aligned}$$

$$\begin{aligned} x_2 + \tfrac{1}{2}x_3 + \tfrac{1}{2}x_4 &= k/2 \quad (1) \\ x_1 + \tfrac{1}{2k}x_3 - \tfrac{1}{2k}x_4 &= \tfrac{1}{2} \quad (2) \end{aligned}$$

$x_i \geq 0$

Gomory cut from (2):

$$\frac{1}{2k}x_3 + \frac{2k-1}{2k}x_4 \geq \frac{1}{2} \quad (3) \Rightarrow x_3 + (2k-1)x_4 \geq k.$$

Substituting for x_3, x_4:

$$k - kx_1 - x_2 + (2k-1)kx_1 - (2k-1)x_2 \geq k$$

$$\Rightarrow \quad \cancel{2k^2x_1 - 2kx} \quad (2k-2)kx_1 - 2kx_2 \geq 0$$

$$\Rightarrow \quad (k-1)x_1 - x_2 \geq 0$$

, as we found earlier.

If use strong Gomory from (2):

$$x_1 \cancel{+ \frac{1}{2k}x_3} + \left(-1 + \frac{\frac{2k-1}{2k} - \frac{1}{2}}{\frac{1}{2}}\right) x_4 \le 0$$

$$\Rightarrow \quad x_1 \cancel{+ \frac{1}{2k}x_3} + \left(-1 + \frac{k-1}{k}\right) x_4 \le 0$$

$$\Rightarrow \quad x_1 \cancel{+ \frac{1}{2k}x_3} - \frac{1}{k} x_4 \le 0$$

Substituting for x_3, x_4:

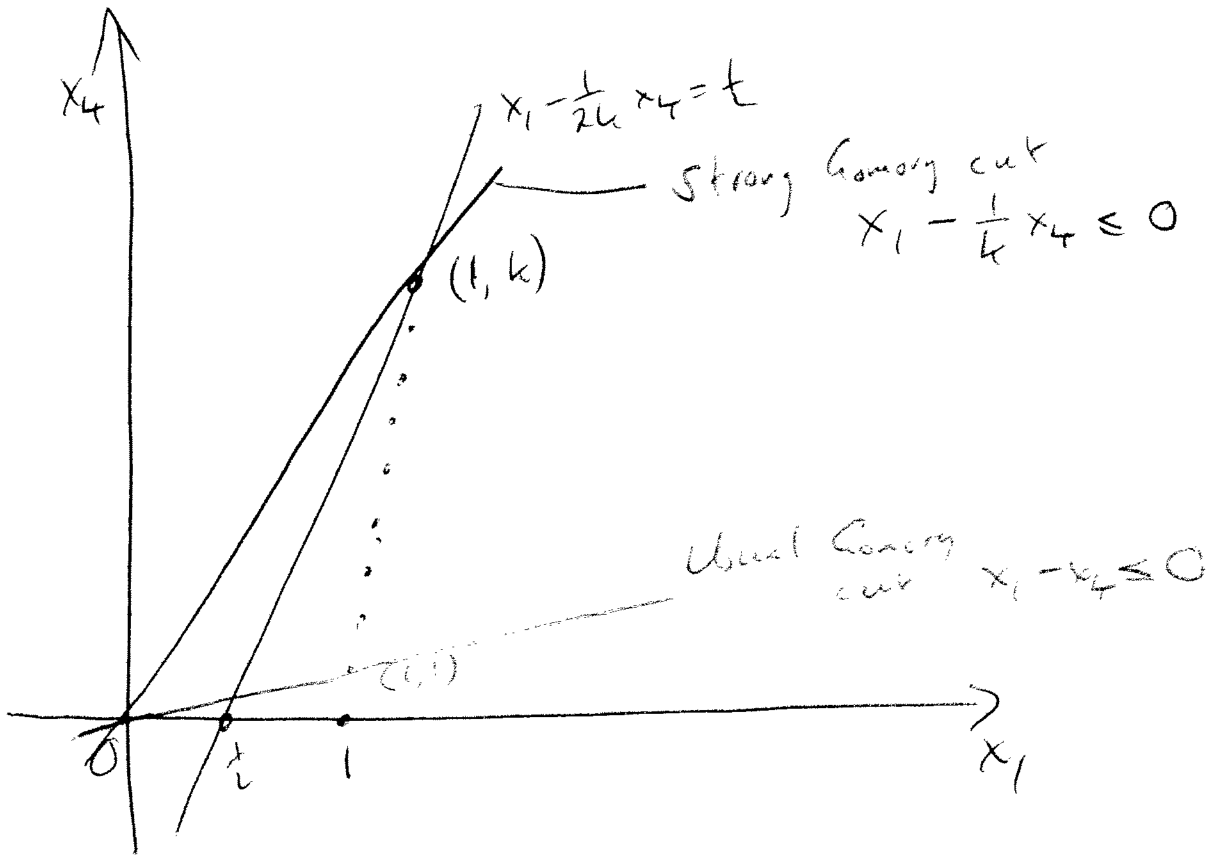
$$x_1 \cancel{+ \frac{1}{2k}(\ldots)} - \frac{1}{k}(kx_1 - x_2) \le 0$$

$$\Rightarrow \quad x_1 \cancel{+ \ldots} - x_1 + \frac{x_2}{k} \le 0$$

$$\Rightarrow \quad \boxed{x_2 \le 0}.$$

Graphically:

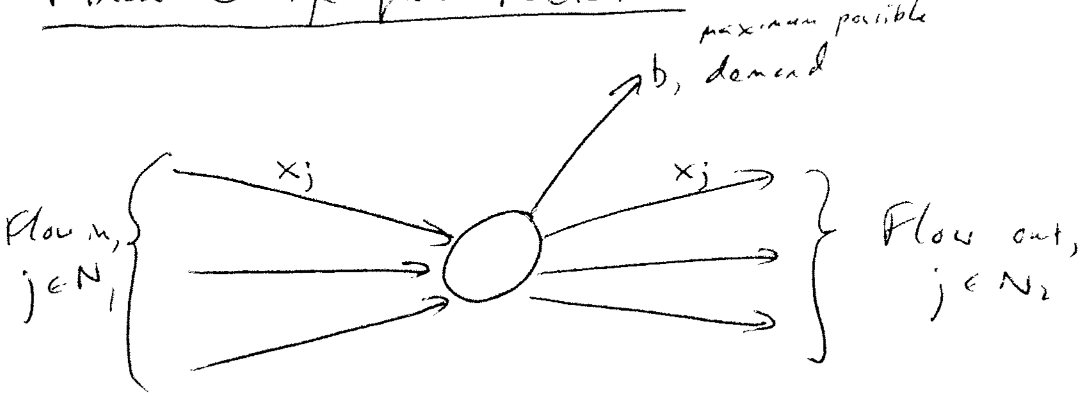
(2) can be weakened to $x_1 - \frac{1}{2k} x_4 \le \frac{1}{2}$



Any C-G cut takes a valid constraint (eg. $x_1 - x_4 \le \frac{1}{2}$) and pushes it until it hits an integer point (eg $x_1 = 1, x_4 = 1$), even if that point violates the original constraint.

The Strong Gomory Cut ignores the integrality of x_4, anchors the current cut where it crosses $x_1 = 1$, and pushes until it hits an integer point with $x_1 = 0$.

Fixed Charge Flow Network:



Flow conservation:

$$\sum_{j \in N_1} x_j \le b + \sum_{j \in N_2} x_j$$

Pay a cost c_j for using an arc, and the arc has capacity a_j.

So use 0-1 variable y_j: $x_j \le a_j y_j$,

place term $c_j y_j$ in objective function.

b and a_j may be fractional, so x may also be fractional.

Flow cover inequalities:

Defn A set $C = C_1 \cup C_2$ with $C_1 \subseteq N_1$, $C_2 \subseteq N_2$ is a GENERALIZED COVER for X if $\sum_{j \in C_1} a_j - \sum_{j \in C_2} a_j = b + \lambda$ for some $\lambda > 0$.

Notice that we can't have $x_j = \begin{cases} a_j & \forall j \in C_1 \cup C_2 \\ 0 & \text{otherwise} \end{cases}$.

If we have a cover, the inequality

$$\sum_{j \in C_1} x_j + \sum_{j \in C_1} (a_j - \lambda)^+ (1 - y_j) \le b + \sum_{j \in C_2} a_j + \lambda \sum_{j \in L_2} y_j + \sum_{j \in N_2 \setminus (C_2 \cup L_2)} x_j$$

[illegible]

is valid, for any $L_2 \subseteq N_2 \setminus C_2$. (Wolsey, Prop 9.6)
([illegible])

Separation routines to find such ineqs which are violated are described in Wolsey, Eg: $x_1 + x_2 + x_3 \le 3 + x_4 + x_5$

$x_1 \le 2y_1$, $x_2 \le y_2$, $x_3 \le 6y_3$, $x_4 \le 3y_4$, $x_5 \le 5y_5$

① $C_1 = \{1, 3\}$, $C_2 = \{4\}$, $\lambda = 2$.

Get

$$x_1 + x_3 + (2-2)^+(1-y_1) + (6-2)^+(1-y_3) \le 3 + 3 + \underbrace{\cancel{2(y_4)}\ 2y_5}_{\text{if } L_2 = \{5\}}$$

Could replace this with

x_5 $(\lambda = 4)$

② i.e., $x_1 + x_3 + 0(1-y_1) + 4(1-y_3) \le \cancel{3}\,6 + 2y_5$

i.e., $x_1 + x_3 - 4y_3 \le \cancel{3}\,2 + 2y_5$ (1)

Seems reasonable, since $x_1 = 2$, $x_3 = 6$ (i.e., at their limits), then forces $y_5 = 1$. (rather than $y_5 = \frac{2}{5}$ with $x_5 = 2$)

Note that this is infeasible, as shown by the version with x_5:

$$x_1 + x_3 - y_1 - 4y_3 \le 1 + x_5 \quad (2)$$

x_1	x_3	y_1	y_3	x_5
2	6	1	1	1

With $x_1 = 2$, $x_3 = 6$, have lhs ≥ 3. But RHS ≤ 2.

If change bound on x_5 to $x_5 \le 3y_5$, (1) & (2) are not changed, but we can now satisfy (2).