MIXED INTEGER NONLINEAR PROGRAMMING

(Survey: Grossmann, Optim & Eng 3 (2002) pp 227–252.)

$$\min \; f(x,y)$$
$$\text{s.t.} \; g_i(x,y) \le 0 \;\; \forall i \in I \qquad \text{(MINLP)}$$
$$x \in X, \; y \in Y$$

Assume X denotes a continuous set of variables and Y denotes a set of variables constrained to be integer.
I = set of constraints.

Arises in many engineering applications, and elsewhere.

Hard to solve computationally: far harder than pure ILP or MILP.

Various approaches motivated by relaxations and approximations to (MINLP)

First, take ~~for~~ some simplifications:

(i) functions linear in y.
(ii) y are binary, not general integers
(iii) functions are convex ⇒

So (abusing notation)

$$\min \; f(x) + c^T y$$
$$\text{s.t.} \; g_i(x) + a_i^T y \le b_i \qquad \text{(MINLP)}$$
$$x \in X, \; y \in Y$$

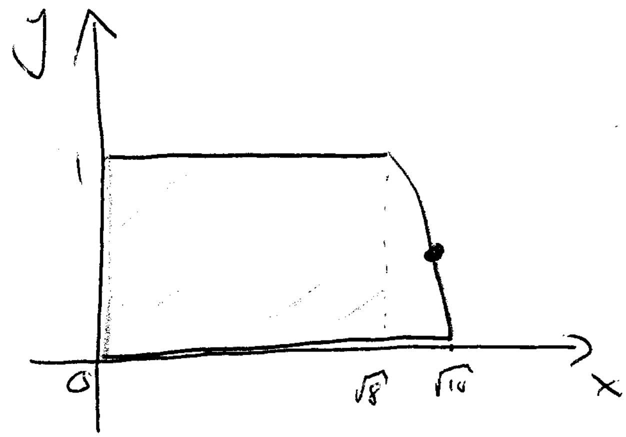
f, g_i convex, $Y \subseteq \mathbb{B}^p$, $X \subseteq \mathbb{R}^n$.

Linear in y, convex in x:

$$\min \ \tfrac{1}{2}(x-9)^2 - 2y$$

$$\text{s.t.} \ \tfrac{1}{2}x^2 + y \leq 5$$

$$0 \leq x \leq 5, \quad y \text{ binary}$$



Optimal soln to NLP relaxation: $x = 3$, $y = \tfrac{1}{2}$.

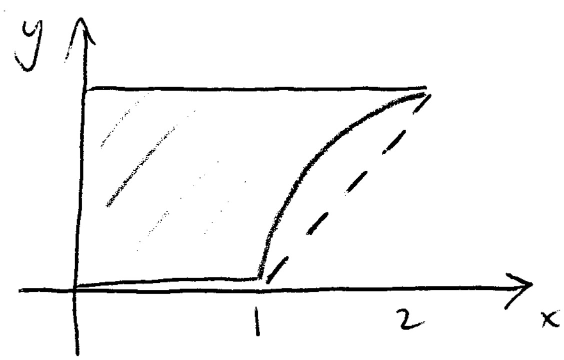
So need to branch.

Linear in y, concave in x:

$$\min \ -\tfrac{1}{2}x^2 + 2y$$

$$\text{s.t.} \ -(x-2)^2 - y \leq -1$$

$$0 \leq x \leq 2, \quad y \text{ binary}$$



Two local minimizers to NLP relaxation: (i) $x=1$, $y=0$
(ii) $x=2$, $y=1$.

Both are feasible for original problem.
Not enough to just find one of the points.

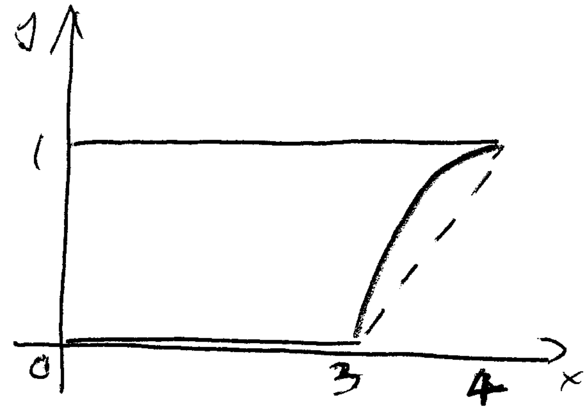
Replace constraint by convex envelope
$+x - y \leq 1$.
Now $x=2, y=1$ is no longer a local min.
This technique might not always work.

Linear in y, concave in x

$$\min \; -\tfrac{1}{2}(x-1)^2 + 2y$$

$$\text{s.t.} \; -(x-4)^2 - y \le -1$$

$$0 \le x \le 4, \quad y \text{ binary}$$



~~All of these~~ Three extreme points are local minimizers to the NLP relaxation. ($x=0$, $y=1$ is not a KKT point.)

$x=0,\ y=0$: value $= -\frac{1}{2}$
$x=3,\ y=0$: value $= -2$
$x=4,\ y=1$: value $= -2\frac{1}{2}$

Even ~~calling~~ using the convex envelope, so replace the constraint

by $\quad x - y \le 3$:

The three local mins are still KKT points.

KKT conditions:

$$-x+1 - 2(x-4)\cdot\lambda_1 + \lambda_2 - \lambda_3 = 0$$
$$2 - 1\cdot\lambda_1 + \lambda_4 - \lambda_5 = 0$$

$$\lambda_1\left(-(x-4)^2 - y + 1\right) = 0, \quad \lambda_2(4-x) = 0, \quad \lambda_3(x) = 0$$
$$\lambda_4(1-y) = 0 \quad \lambda_5\, y = 0$$

<u>If fix $y = \bar{y} \in Y$</u>

Get overestimate for optimal value by finding a feasible solution to (and ideally solving!):

$$\begin{aligned} \min_x \quad & f(x, \bar{y}) \\ \text{s.t.} \quad & g_i(x, \bar{y}) \le 0 \quad \forall i \in I \qquad (NLPU) \\ & x \in X \end{aligned}$$

If the functions are convex in x, this is a convex nonlinear program, so can use, eg. interior point method or SQP to solve it.

This subproblem can be used in a generalized Benders decomposition approach (see Grossmann survey, or work of FLOUDAS).

(NLPU) is also useful within a branch-and-bound approach: a good upper bound will enable pruning of some nodes.

Relaxations

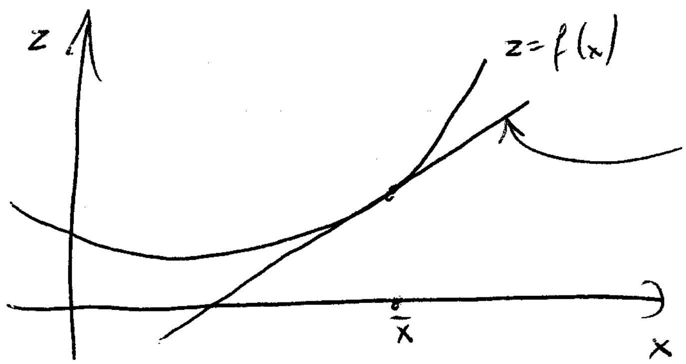
(I) Outer Approximation

Assume f and each g_i are convex functions.

So, given $\bar{x} \in X$, $\bar{y} \in Y$, have:

$$f(x,y) \geq f(\bar{x},\bar{y}) + \nabla f(\bar{x},\bar{y})^T \begin{pmatrix} x - \bar{x} \\ y - \bar{y} \end{pmatrix}$$

$$g_i(x,y) \geq g_i(\bar{x},\bar{y}) + \nabla g_i(\bar{x},\bar{y})^T \begin{pmatrix} x - \bar{x} \\ y - \bar{y} \end{pmatrix}$$

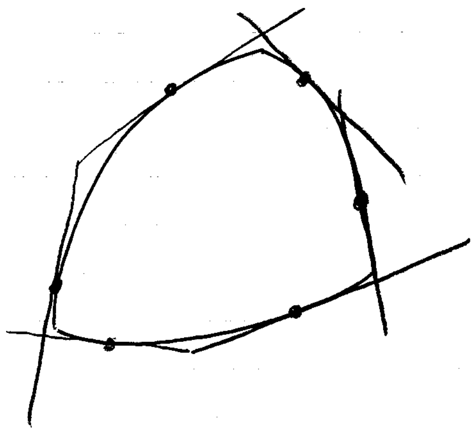


Can replace the convex functions by piecewise linear approximations:

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & z \geq f(x^j,y^j) + \nabla f(x^j,y^j)^T \begin{pmatrix} x - x^j \\ y - y^j \end{pmatrix} \quad \forall j \in J \\ & g_i(x^j,y^j) + \nabla g_i(x^j,y^j)^T \begin{pmatrix} x - x^j \\ y - y^j \end{pmatrix} \leq 0 \quad \forall i \in I,\ \forall j \in J \\ & x \in X,\ y \in Y \end{aligned} \qquad \text{(OASP)}$$

where $\{(x^j,y^j) : j \in J\}$ is a set of points in $X \times Y$.

Approximation to feasible region



Problem (OASP) is a mixed integer linear program. The solution to (OASP) gives a lower bound on the optimal value of (MINLP)

Algorithm uses the following loop:

(1) Given a set J of points, solve the MILP (OASP). Get $\bar{x}, \bar{y}$.

(2) Solve (NLPU) with $\bar{y}$ to get an appropriate $\tilde{x}$

(3) Add new constraints to (OASP) using $x^j = \tilde{x}$, $y^j = \bar{y}$.

Step (1) gives a lower bound, and Step (2) gives an upper bound.

II) Branch-and-Bound

Relax $y \in Y$ to a continuous domain $\bar{Y}$, as in B&B for mixed integer linear programming.

$$\min \ f(x,y)$$
$$\text{s.t.} \ g_i(x,y) \leq 0 \quad \forall i \in I \qquad \text{(NLP)}$$
$$x \in X, \ y \in \bar{Y}$$

Assume functions are all convex.

Then any local minimizer to (NLP) is also a global minimizer.

Typical algorithms for (NLP) converge to a local minimizer. (Eg interior point, SQP, ...)

Also, can get lower bounds on this problem ~~B~~ using Lagrangian duality.

~~Eg.~~ For problems near root of tree, often sufficient to solve them approximately:

(a) If could prune by bounds, or infeasibility, can usually detect fairly quickly

(b) If will need to ~~split~~ ~~the~~ branch on the node, don't need a very accurate solution

(c) If ~~n~~ soln is feasible in (MINLP), then should try to solve (NLP) to optimality.

Efficiently warm starting the child node is an open problem.

~~E~~ Interior point methods work best when started from reasonably central points, so ~~start~~ warm start from earlier iterates ~~B~~ of the parent.

Need a robust solver for the NLP subproblems, since need to be confident that don't determine (i) a feasible node is infeasible, or (ii) an invalid lower bound.

Eg: Tables 2, 3, 4 from "Solving Large MINLPs on Computational Grids", Goux & Leyffer, Optimization & Engineering 3, 327-346, 2002.

Solve problems with no more than ~~1000~~ ~~$\sim 10^3$~~ 750 integer variables. ~~Requires large effort~~

Requires from $\sim 10^3 - 10^9$ nodes in B&B tree.

Run time measured as a few hours on a computational grid (up to 146 nodes running simultaneously)

This paper solves nonconvex problems also, by using heuristic methods to get a lower bound.

Nonconvex problems

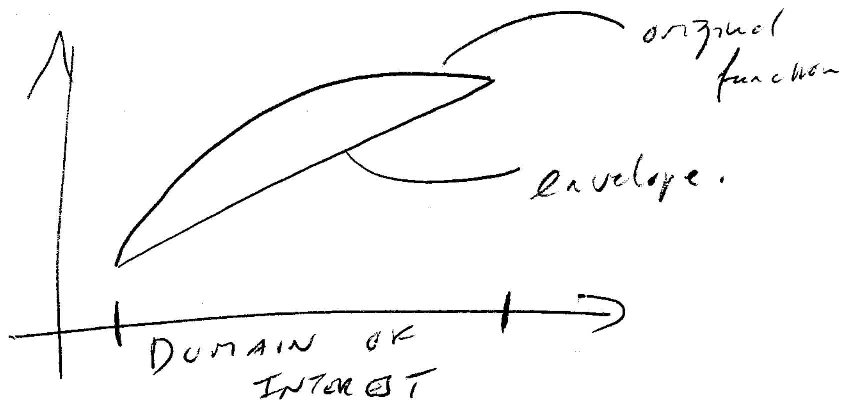
Eg: Tawarmalani & Sahinidis, MathProg 99 (2004), pp 563-591.

Branch on nonconvexity to get global optimum,

in addition to branching on ~~variables~~ integer variables.

So partition the feasible region.

Put envelopes on functions to get lower bounds:



The reference states: (page 565, Remark 2):

"Dropping integrality conditions may not suffice to obtain a useful relaxation of an MINLP. Indeed, the resulting MINLP may involve nonconvexities that make solution difficult. It is even possible that the resulting NLP may be harder to solve to global optimality than the original MINLP."

Leyffer states in his 2001 COAP paper:

" A large number of NLP ~~relaxation~~ subproblems often have no physical meaning if the integer variables do not take integer values."

Tawarmalani & Sahinidis look especially at nonconvex problems, and ~~they~~ ~~fo~~, so they give results for small problems (< 100 vars in almost all cases). They investigate especially concave quadratic problems.

TESTSETS:

(1) MINLP LIB. http://www.gamsworld.org/minlp/minlplib.htm

(2) MacMINLP: http://www-unix.mcs.anl.gov/~leyffer/macminlp

Some overlap (eg trimloss in (2) ↔ tls in (1)).