

LIFT-AND-PROJECT INEQS. (WOLSEY, §8.8) (ALL DISJUNCTIVE INEQUALITIES)

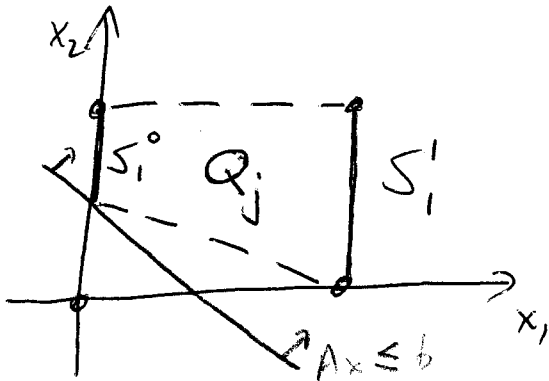
$$S = \{x \in B^n : Ax \leq b\}.$$

$$S_j^0 = \{x \in B^n : x_j = 0\}, \quad S_j^1 = \{x \in B^n : x_j = 1\}.$$

$Ax \leq b, 0 \leq x \leq 1$                        $Ax \leq b, 0 \leq x \leq 1$

Consider  $Q_j = \text{conv}\{S_j^0 \cup S_j^1\}$

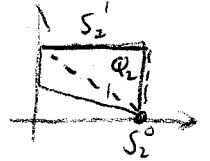
Eg:



If we iterate this process  
(ie, find  $Q_1$ , then define  $Q_2$   
based on  $Q_1$ , etc.)

we get  $\text{conv}(S)$ .

(Balas, 1975).



Valid inequalities for  $Q_j$  are called LIFT-AND-PROJECT inequalities.

Need to solve an LP to find such an inequality:

If  $\pi^T x \leq \pi_0$  is such an inequality, then have to ensure this inequality is valid for both  $S_j^0$  and  $S_j^1$ .

Try to choose strongest such inequality that cuts off a fractional point.

These inequalities have been successfully used in cut-and-branch approaches

to integer programs: Balas, Ceriz, Cornuejols, 1996  
Ceriz, Pataki, 1998.

Now integrated into CPLEX.

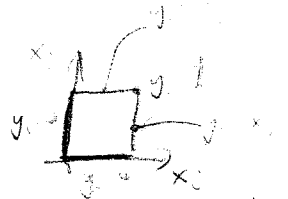
Sequential convexification procedure:

- (i) Select an index  $j$
- (ii) Multiply  $Ax \leq b$  by  $1-x_j, x_j$  to obtain nonlinear system

$$\begin{aligned} (1-x_j)(Ax-b) &\leq 0 & (*) \\ x_j(Ax-b) &\leq 0 \end{aligned}$$

- (iii) Linearize (\*) by substituting  $y_{ij}$  for  $x_i x_j, i \neq j$ , and  $x_j$  for  $x_j^2$ .

- (iv) Project onto the  $x$ -space:  $x$  is feasible if there exists a  $y$  so that  $(x,y)$  is feasible at step (iii).



Step (iii) yields ineqs  $y_{ij} \geq 0, y_{ij} \leq x_i, y_i \geq x_i + x_j - 1$ , plus more ineqs.

Finding the best ineqs in step (iv) requires solving a LP for each ineq.

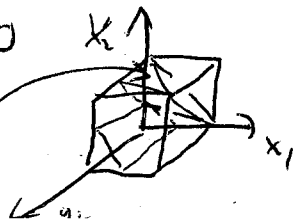
Repeating this process for all indices gives the convex hull.

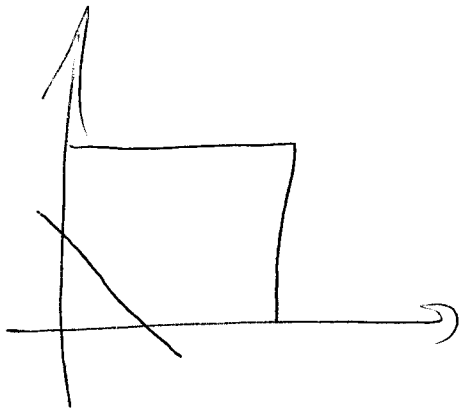
Go through example:  $-2x_1 - 2x_2 \leq -1$ .

Get: multiply by  $x_1$ :  $\Rightarrow -x_1 - 2y_2 \leq 0$  - redundant.  
 multiply by  $(1-x_1)$ :  $\Rightarrow 1-x_1 - 2x_2 + 2y_2 \leq 0$

or  $x_1 + 2y_2 \geq 1 - x_2$   
 $x_1 + x_2 \leq 1 + y_2$   
 $0 < x_1 < x_2$

Get this wedge for linearized problem.





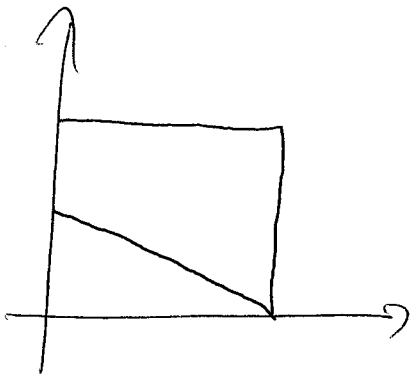
$$2x_1 + 2x_2 \geq 1$$

$$1-x_1 \left( \begin{array}{l} \downarrow x_1 \\ 2x_1 + 2x_2 \geq x_1 \\ \text{or } x_1 + 2x_2 \geq 0 \end{array} \right.$$

$$2x_1 + 2x_2 - 2x_1 - 2x_2 \geq 1 - \cancel{2x_1} x_1$$

$$\text{or } x_1 + x_2 \geq 1 + 2x_2$$

$$\geq 1$$



$$x_1 + 2x_2 \geq 1$$

$$1-x_2 \left( \begin{array}{l} \downarrow x_2 \\ \cancel{x_1 + 2x_2} \\ y_1 + 2x_2 \geq x_2 \text{ or } y_1 + x_2 \geq 0 \end{array} \right.$$

$$x_1 + 2x_2 - y_1 - 2x_2 \geq 1 - x_2$$

$$\text{or } x_1 + x_2 \geq 1 + y_1 \geq 1 \quad \checkmark$$

L8P 3a.

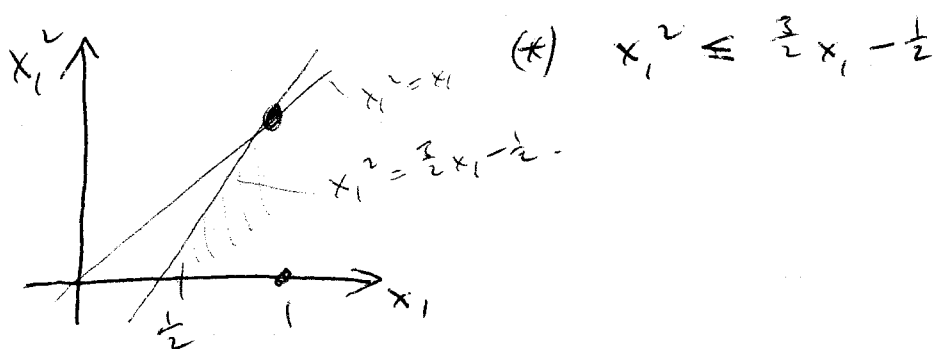
$$x_1 \geq \frac{1}{2} \quad x_1 \text{ binary}$$

$$(1-x_1)(x_1 - \frac{1}{2}) \geq 0$$

$$x_1 - x_1^2 - \frac{1}{2} + \frac{1}{2}x_1 \geq 0 \quad (*)$$

Exploiting  $x_1^2 = x_1$  is crucial.

$$\frac{1}{2}x_1 \geq \frac{1}{2} \quad x_1 \geq 1.$$



$$x_1(x_1 - \frac{1}{2}) \geq 0 \Rightarrow x_1^2 \geq \frac{1}{2}x_1 \quad \text{True for all } x_1 \in [0, 1].$$

# General disjunctions

(Survey paper:  
BACAS & PERREGAARD, Discrete Applied Math,  
2002, Vol 123, pages 129-154)

$$P = \bigcup_{q \in Q} \{x \in \mathbb{R}^n : A^q x \geq b^q\}$$

Note: any simple bounds  
are included in " $A^q x \geq b^q$ ".

Convex hull:

$$\text{conv}(P) = \left\{ x \in \mathbb{R}^n : \begin{array}{l} x - \sum_{q \in Q} y^q = 0 \\ A^q y^q - b^q y_0^q \geq 0 \quad \forall q \in Q \\ y_0^q \geq 0 \quad \forall q \in Q \\ \sum_{q \in Q} y_0^q = 1 \\ \text{for some } (y^q, y_0^q) \in \mathbb{R}^{n+1}, \\ q \in Q \end{array} \right\}$$

Eg: Binary integer program: Disjunction on  $x_1$ :

$$\text{Let } Q^0 = \{x \in \mathbb{R}^n : Ax \geq b, x_1 = 0\}$$

$$Q^1 = \{x \in \mathbb{R}^n : Ax \geq b, x_1 = 1\}$$

$$P = Q^0 \cup Q^1$$

$$\text{conv}(P) = \{x \in \mathbb{R}^n : \exists y^1, y^2 \in \mathbb{R}^n, y_0^1, y_0^2 \in \mathbb{R}_+, \text{ with}$$

$$x - y^1 - y^2 = 0$$

$$Ay^1 - by_0^1 \geq 0, y_1^1 = 0$$

$$Ay^2 - by_0^2 \geq 0, y_1^2 = y_0^2$$

$$y_0^1 + y_0^2 = 1$$

Exactly the set  
obtained after the  
linearization step in  
the sequential convexification  
procedure.

## Finding valid inequalities for $\text{conv}(P)$

Given the expression for  $\text{conv}(P)$ , want to project out the  $y$  variables.

$$=: P_Q$$

Valid constraints for  $\text{conv}(P)$  are all constraints of the form

$$\alpha^T x \geq \beta$$

where:  $\left. \begin{array}{l} \alpha^T = \sum u^q A^q \\ \beta \leq \sum u^q b^q \end{array} \right\}$  for some  $u^q \geq 0$  for each  $q \in Q$

This is the polar cone of  $P_Q$

Want a constraint that is violated by the current point  $\bar{x}$ .

$$\begin{array}{ll} \min_{\alpha, \beta, u^q} & \alpha^T \bar{x} - \beta \\ \text{s.t.} & \alpha - \sum A^q u^q = 0 \quad q \in Q \\ & \beta - \sum b^q u^q \leq 0 \quad q \in Q \\ & u^q \geq 0 \quad q \in Q \end{array}$$

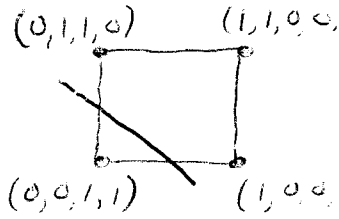
Since the feasible region is a cone, add a NORMALIZATION constraint. Eg:

$$\sum_{q \in Q} e^T u^q = 1 \quad e = \text{vector of ones.}$$

LOP 5a

Eg:  $x_1 + x_3 = 1$   
 $x_2 + x_4 = 1$   
 $2x_1 + x_2 \geq 1$   
 $x_i \geq 0$

Disjunction:  $x_1 \perp x_3, x_2 \perp x_4$ .



Feasible point to relaxation:  $\bar{x} = (\frac{1}{2}, 0, \frac{1}{2}, 1)$ .

Three feasible points:  
 $(0, 1, 1, 0), (1, 1, 0, 0), (1, 0, 0, 1)$ .

Subproblem is:

min  $\frac{1}{2}x_1 + \frac{1}{2}x_3 + x_4 - \beta$

s.t.  $x \geq \begin{bmatrix} 1 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} u^1 \Rightarrow (x_i \leq 0)$

Inequality, since have  $x_i \geq 0$ .

$x \geq \begin{bmatrix} 1 & -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} u^2 \Rightarrow (x_3 \leq 0)$

$\beta \leq [1 \ -1 \ 1 \ -1 \ 1 \ 0] u^1$

$\beta \leq [1 \ -1 \ 1 \ -1 \ 1 \ 0] u^2$

$e^T u^1 + e^T u^2 \leq 1$

$u^1, u^2 \geq 0$ .

Soln:  $\alpha = (0, \frac{1}{2}, -\frac{1}{4}, 0), \beta = 0$ .

Constraint:  $\frac{1}{2}x_2 - \frac{1}{4}x_3 \geq 0$ . Violated by  $\bar{x}$ , satisfied by three feasible points

$u^1 = (0, \frac{1}{4}, 0, 0, \frac{1}{4}), u^2 = (0, 0, 0, 0, 0, \frac{1}{4})$

(Found using AMPL)

LIFT-AND-PROJECT EXAMPLE

$$S = \{x \in \mathbb{B}^2 : 2x_1 + 2x_2 \geq 1\}$$

Want to cut off the fractional point  $(\frac{1}{2}, 0)$

Set up subproblem:

$$\min \frac{1}{2} \alpha_1 - \beta$$

$$\text{s.t. } \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & -1 & 0 & -1 \\ 2 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} u^1$$

$$\begin{array}{l} \text{Since} \\ S^0 = \{x \in \mathbb{R}^2 : 2x_1 + 2x_2 \geq 1 \\ x_1 \geq 0 \\ x_2 \geq 0 \\ -x_1 \geq -1 \\ -x_2 \geq -1 \\ -x_1 \geq 0 \} \end{array}$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & -1 & 0 & 1 \\ 2 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} u^2$$

$$\begin{array}{l} S^1 = \{x \in \mathbb{R}^2 : 2x_1 + 2x_2 \geq 1 \\ x_1 \geq 0 \\ x_2 \geq 0 \\ -x_1 \geq -1 \\ -x_2 \geq -1 \\ x_1 \geq 1 \} \end{array}$$

$$\beta \leq [1 \ 0 \ 0 \ -1 \ -1 \ 0] u^1$$

$$\beta \leq [1 \ 0 \ 0 \ -1 \ -1 \ 1] u^2$$

$$e^T u^1 + e^T u^2 \leq 1 \\ u^1, u^2 \geq 0$$

Soln:  $\alpha_1 = 0.2, \alpha_2 = 0.4, \beta = 0.2, u^1 = (0.2, 0, 0, 0, 0, 0), u^2 = (0, 0, 0.4, 0, 0, 0)$

Given the constraint  $0.2x_1 + 0.4x_2 \geq 0.2$   
ie,  $x_1 + 2x_2 \geq 1$

In practice:

As presented, the cut generation LP is large.

So: (i) Only include columns of  $A^2$  corresponding to basic variables.

Can lift later to include remaining variables.

This is OK theoretically, nothing is lost

(ii) Only include rows of  $A^2$  corresponding to active constraints.

Resulting cuts might be weaker.

Multiple cuts can be generated from a disjunction, by using various pivots:

Eg: change the  $\bar{x}$ ,

and solve a slightly different cut generation LP.

Need to combine the cut generation with branching.

Cut strengthening

The disjunction  $\sum_{j \in R} g_j x_j \geq 1 \quad \vee \quad \sum_{j \in R} h_j x_j \geq 1$

leads to the cut

$$\sum_{j \in R} \alpha_j x_j \geq 1 \quad \text{with } \alpha_j = \max\{g_j, h_j\}.$$

In the integer case, this can be strengthened using a closed form expression:

If for some  $j$ , we have  $g_j \gg h_j$ , would like to reduce  $g_j$  and increase  $h_j$ .

May also be possible even if  $x_j$  need not be integer.

But no closed form expression.

A valid cut is feasible in the cut generation LP.

This LP gives a systematic way to strengthen the cuts, by optimizing in the LP.

## Other Procedures

### ① Lovasz & Schrijver

Have inequalities  $\tilde{A}x \geq \tilde{b}$  (including all bounds).

Sequential convexification / lift-and-project multiplies this

system by  $x_j$  and  $1-x_j$  for one particular  $x_j$ ,

then linearizes, and repeats for other  $x_j$ .

Lovasz & Schrijver multiply  $\tilde{A}x \geq \tilde{b}$  by every  $x_j$  and  $1-x_j$ .

Still need to repeat the process  $p$  times if have

$p$  integer variables, but get = benefits coefficient

matrix of linearized system is positive definite.

### ② Sherali & Adams

Multiply by products of the form  $\left(\prod_{j \in J_1} x_j\right) \left(\prod_{j \in J_2} (1-x_j)\right)$

for some disjoint subsets  $J_1, J_2$ .

Then linearize.

Multiple applications will give the convex hull.

A single application may give something useful.