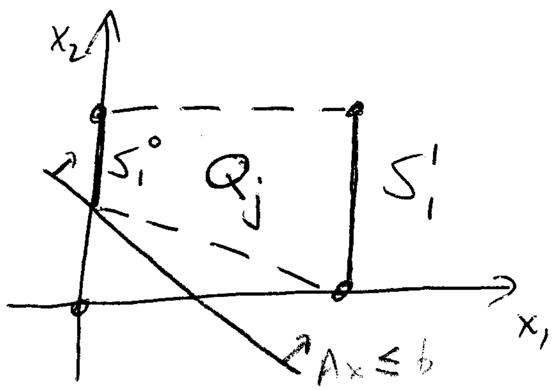
LIFT-AND-PROJECT INEQS. (WOLSEY, §8.8) (aka DISJUNCTIVE INEQUALITIES)

$S = \{x \in B^n : Ax \le b\}$.

$S_j^0 = \{x : x_j = 0\}$, $S_j^1 = \{x : x_j = 1\}$.
$\quad Ax \le b, 0 \le x \le e \qquad\qquad Ax \le b, 0 \le x \le e$

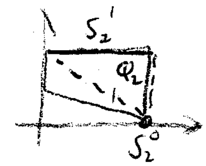
Consider $Q_j = \text{conv}\{S_j^0 \cup S_j^1\}$

Eg:



If we iterate this process (ie, find Q_1, then define Q_2 based on Q_1, etc.), we get ~~=~~ conv(S).

(Balas, 1975).



Valid inequalities for Q_j are called LIFT-AND-PROJECT inequalities.

Need to solve an LP to find such an inequality:

If $\pi^T x \le \pi_0$ is such an inequality, then have to ensure this ineq is valid for both S_j^0 and S_j^1.

Try to choose strongest such ineq that cuts off a fractional point.

These inequalities have been successfully used in cut-and-branch approaches to integer programs: Balas, Ceria, Cornuejols, 1996
Ceria, Pataki, 1998.

Now integrated into CPLEX.

Sequential convexification procedure:

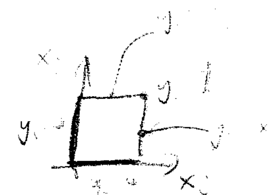
(i) Select an index j

(ii) Multiply $Ax \le b$ by $1-x_j$, x_j to obtain nonlinear system

$$\begin{aligned} (1-x_j)(Ax-b) &\le 0 \qquad (*) \\ x_j\,(Ax-b) &\le 0 \end{aligned}$$

(iii) Linearize $(*)$ by substituting y_{ij} for $x_i x_j$, $i \ne j$, and x_j for x_j^2

(iv) Project onto the x-space: x is feasible if there exists a y so that (x,y) is feasible at step (iii).



Step (iii) yields ineqs $y_{ij} \ge 0$, $y_{ij} \le x_i$, $y_{ij} \ge x_i + x_j - 1$, plus more ineqs.

Finding the best ineqs in step (iv) requires solving an LP for each ineq.

Repeating this process for all indices gives the convex hull.

Go through example: $-2x_1 - 2x_2 \le -1$.

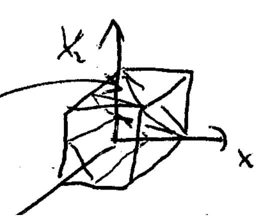
Get: multiply by x_1: $\Rightarrow$ $-x_1 - 2y_2 \le 0$ - redundant.

multiply by $(1-x_1)$: $\Rightarrow$ $1 - x_1 - 2x_2 + 2y_2 \le 0$

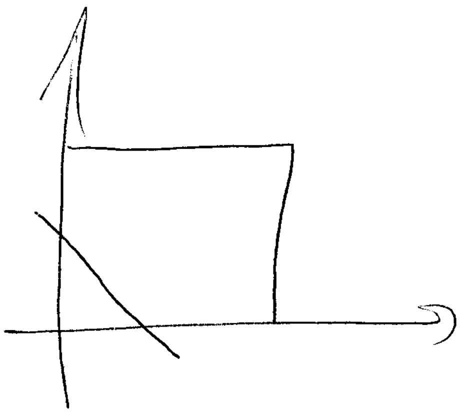
or $x_1 + 2x_2 \ge 1 + 2y_2$

$x_1 + x_2 \le 1 + y_2$

Get this wedge for linearized



L8 P3



$$2x_1 + 2x_2 \geq 1$$

$\xrightarrow{x_1}$

$$2x_1 + 2y_2 \geq x_1$$

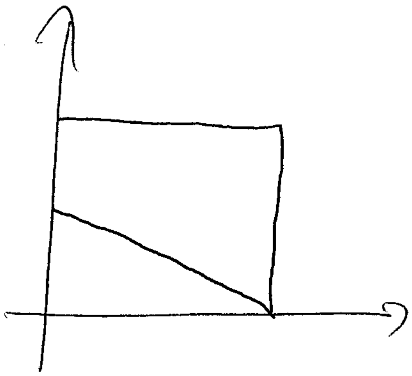
or $x_1 + 2y_2 \geq 0$

$\xrightarrow{1-x_1}$

$$2x_1 + 2x_2 - 2x_1 - 2y_2 \geq 1 - \cancel{2x_2}\, x_1$$

or $x_1 + 2x_2 \geq 1 + 2y_2$

≥ 1



$$x_1 + 2x_2 \geq 1$$

$\xrightarrow{x_2}$

$\cancel{x_1 + 2y_1}$

$y_1 + 2x_2 \geq x_2$ or $y_1 + x_2 \geq 0$

$\xrightarrow{1-x_2}$

$$x_1 + 2x_2 - y_1 - 2x_2 \geq 1 - x_2$$

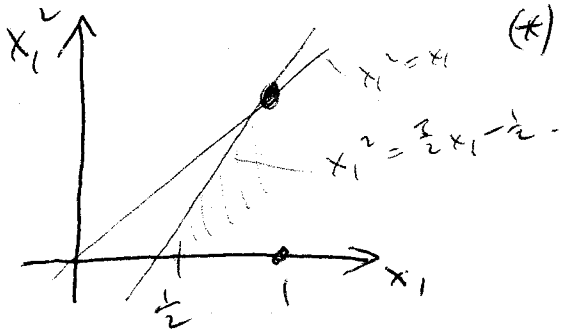
or $x_1 + x_2 \geq 1 + y_1 \geq 1$ ✓.

$x_1 \geqslant \frac{1}{2}$ $\quad$ x_1 binary

$(1-x_1)(x_1 - \frac{1}{2}) \geqslant 0$

$x_1 - x_1^2 - \frac{1}{2} + \frac{1}{2}x_1 \geqslant 0$ $\quad (*)$ $\qquad$ Exploiting $x_1^2 = x_1$ is crucial.

$\frac{1}{2}x_1 \geqslant \frac{1}{2}$ $\quad$ $x_1 \geqslant 1$.



$(*)$ $\quad x_1^2 \leqslant \frac{3}{2}x_1 - \frac{1}{2}$

$x_1(x_1 - \frac{1}{2}) \geqslant 0 \Rightarrow x_1^2 \geqslant \frac{1}{2}x_1$ $\quad$ True for all $x_1 \in [0,1]$.

General disjunctions

(Survey paper: BALAS & PERREGAARD, Discrete Applied Math 2002, Vol 123, pages 129–154.)

$$P = \bigcup_{q \in Q} \{x \in \mathbb{R}^n : A^q x \geq b^q\}$$

Note: any simple bounds are included in "$A^q x \geq b^q$".

Convex hull:

$$\operatorname{conv}(P) = \left\{ x \in \mathbb{R}^n : \begin{array}{ll} x - \sum_{q \in Q} y^q = 0 & \\ A^q y^q - b^q y_0^q \geq 0 & \forall q \in Q \\ y_0^q \geq 0 & \forall q \in Q \\ \sum_{q \in Q} y_0^q = 1 & \\ \text{for some } (y^q, y_0^q) \in \mathbb{R}^{n+1}, & q \in Q \end{array} \right\}$$

Eg: Binary integer program: Disjunction on x_1:

Let $Q^0 = \{x \in \mathbb{R}^n : Ax \geq b,\ x_1 = 0\}$

$Q^1 = \{x \in \mathbb{R}^n : Ax \geq b,\ x_1 = 1\}$

$P = Q^0 \cup Q^1$

$$\operatorname{conv}(P) = \left\{ x \in \mathbb{R}^n : \exists\, y^1, y^2 \in \mathbb{R}^n,\ y_0^1, y_0^2 \in \mathbb{R}_+, \text{ with } \begin{array}{l} x - y^1 - y^2 = 0 \\ Ay^1 - by_0^1 \geq 0,\ y_1^1 = 0 \\ Ay^2 - by_0^2 \geq 0,\ y_1^2 = y_0^2 \\ y_0^1 + y_0^2 = 1 \end{array} \right\}$$

Exactly the set obtained after the linearization step in the sequential convexification procedure.

Finding valid inequalities for conv(P)

Given the expression for $\underbrace{\text{conv}(P)}_{=:P_Q}$, want to project out the y variables.

Valid constraints for conv(P) are all constraints of the form

$$\alpha^T x \geq \beta$$

where: $\left.\begin{array}{l} \alpha^T = u^{q^T} A^q \\ \beta \leq u^{q^T} b \end{array}\right\}$ for some $u^q \geq 0$ for each $q \in Q$

This is the polar cone of P_Q

Want a constraint that is violated by the current point $\bar{x}$.

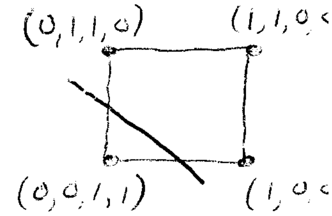
$$\begin{array}{llll} \min\limits_{\alpha,\beta,u^q} & \alpha^T \bar{x} - \beta & & \\ \text{s.t.} & \alpha - A^{q^T} u^q & = 0 & q \in Q \\ & \beta - b^T u^q & \leq 0 & q \in Q \\ & u^q \geq 0 & & q \in Q \end{array}$$

Since the feasible region is a cone, add a NORMALIZATION constraint. Eg:

$$\sum_{q \in Q} e^T u^q = 1 \qquad e = \text{vector of ones.}$$

Eg: $$\begin{aligned} x_1 \quad\quad + x_3 \quad\quad &= 1 \\ x_2 \quad\quad + x_4 &= 1 \\ 2x_1 + 2x_2 \quad\quad &\geq 1 \\ x_i &\geq 0 \end{aligned}$$

Disjunction: $x_1 \perp x_3$, $x_2 \perp x_4$.



Three feasible points: $(0,1,1,0)$, $(1,1,0,0)$, $(1,0,0,1)$.

Feasible point to relaxation: $\bar{x} = (\frac{1}{2}, 0, \frac{1}{2}, 1)$.

Subproblem is:

$$\min \ \tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_3 + \alpha_4 - \beta$$

$$\text{s.t.} \quad \alpha \geq \begin{bmatrix} 1 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} u^1 \quad (x_1 \leq 0)$$

Inequality, since have $x_i \geq 0$.

$$\alpha \geq \begin{bmatrix} 1 & -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} u^2 \quad (x_3 \leq 0)$$

$$\beta \leq [1 \ -1 \ 1 \ -1 \ 1 \ 0]\, u^1$$

$$\beta \leq [1 \ -1 \ 1 \ -1 \ 1 \ 0]\, u^2$$

$$e^T u^1 + e^T u^2 \leq 1$$

$$u^1, u^2 \geq 0.$$

Soln: $\alpha = (0 \ \frac{1}{2} \ -\frac{1}{4} \ 0)$, $\beta = 0$.

Constraint: $\frac{1}{2}x_2 - \frac{1}{4}x_3 \geq 0$. Violated by $\bar{x}$, satisfied by three feasible points

$u^1 = (0, \frac{1}{4}, 0, 0, \frac{1}{4})$, $u^2 = (0, 0, 0, 0, 0, \frac{1}{4})$

(Found using AMPL)

Lift-and-Project Examples

$S = \{x \in \mathbb{B}^2 : 2x_1 + 2x_2 \geq 1\}$

Want to cut off the fractional point $(\frac{1}{2}, 0)$

Set up subproblem:

$$\min \tfrac{1}{2}\alpha_1 \quad - \beta$$

$$\text{s.t.} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & -1 & 0 & -1 \\ 2 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} u^1$$

Since

$$S^0 = \{x \in \mathbb{R}^2 : \begin{aligned} 2x_1 + 2x_2 &\geq 1 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \\ -x_1 &\geq -1 \\ -x_2 &\geq -1 \\ -x_1 &\geq 0 \end{aligned}$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & -1 & 0 & 1 \\ 2 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} u^2$$

$$S^1 = \{x \in \mathbb{R}^2 : \begin{aligned} 2x_1 + 2x_2 &\geq 1 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \\ -x_1 &\geq -1 \\ -x_2 &\geq -1 \\ x_1 &\geq 1 \end{aligned}$$

$$\beta \leq [1 \;\; 0 \;\; 0 \;\; -1 \;\; -1 \;\; 0]\, u^1$$

$$\beta \leq [1 \;\; 0 \;\; 0 \;\; -1 \;\; -1 \;\; 1]\, u^2$$

$$e^T u^1 + e^T u^2 \leq 1$$

$$u^1, u^2 \geq 0$$

Soln: $\underbrace{\alpha_1 = 0.2,\ \alpha_2 = 0.4,\ \beta = 0.2}_{\text{Gives the constraint}}$, $u^1 = (0.2, 0, 0, 0, 0, 0.2)$, $u^2 = (0, 0, 0.4, 0, 0,$

Gives the constraint $\quad 0.2x_1 + 0.4x_2 \geq 0.2$

ie, $\quad x_1 + 2x_2 \geq 1$

<u>In practice:</u>

As presented, the cut generation LP is large.

So: (i) Only include columns of A^q corresponding to basic variables.

Can lift later to include remaining variables.

This is OK theoretically, nothing is lost

(ii) Only include rows of A^q corresponding to active constraints.

Resulting cuts might be weaker.

Multiple cuts can be generated from a disjunction, by using various pivots:

Eg: change the $\bar{x}$,

and solve a slightly different cut generation LP.

Need to combine the cut generation with branching.

From simplex tableau

$$x_k + \sum_{j \in J} \bar{a}_{kj} x_j = \bar{a}_{k0}, \quad x_k \text{ binary}, \quad 0 < \bar{a}_{k0} < 1$$

So either $\sum_{j \in J} \bar{a}_{kj} x_j \geq \bar{a}_{k0}$

or $\sum_{j \in J} \bar{a}_{kj} x_j \leq \bar{a}_{k0} - 1$

So either $\sum_{j \in J} \frac{\bar{a}_{kj}}{\bar{a}_{k0}} x_j \geq 1$

or $\sum_{j \in J} \frac{-\bar{a}_{kj}}{1-\bar{a}_{k0}} x_j \geq 1$

So: $$\sum_{j \in J} \max\left\{ \frac{\bar{a}_{kj}}{\bar{a}_{k0}}, \frac{-\bar{a}_{kj}}{1-\bar{a}_{k0}} \right\} x_j \geq 1$$

Same as Gomory Mixed Integer Cuts

when all x_j $(j \in J)$ treated as continuous

Cut strengthening

The disjunction $\sum_{j \in R} g_j x_j \geq 1 \quad \vee \quad \sum_{j \in R} h_j x_j \geq 1$

leads to the cut

$$\sum_{j \in R} \alpha_j x_j \geq 1 \quad \text{with } \alpha_j = \max\{g_j, h_j\}.$$

In the integer case, this can be strengthened using a closed form expression:

If for some j, we have $g_j \gg h_j$, would like to reduce g_j and increase h_j.

May also be possible even if x_j need not be integer.

But no closed form expression.

A valid cut is feasible in the cut generation LP.

This LP gives a systematic way to strengthen the cuts, by optimizing in the CP.

Other Procedures

① Lovasz & Schrijver

Have inequalities $\tilde{A}x \geq \tilde{b}$ (including all bounds).

Sequential convexification / lift-and-project multiplies this system by x_j and $1-x_j$ for one particular x_j, then linearizes, and repeats for other x_j.

Lovasz & Schrijver multiply $\tilde{A}x \geq \tilde{b}$ by <u>every</u> x_j and $1-x_j$.

Still need to repeat the process p times if have p integer variables, but get a benefit: coefficient matrix of linearized system is positive ~~semi~~definite.

② Sherali & Adams

Multiply by products of the form $\left(\prod_{j \in J_1} x_j\right)\left(\prod_{j \in J_2} (1-x_j)\right)$ for some disjoint subsets J_1, J_2.

Then linearize.

Multiple applications will give the convex hull.

A single application may give something useful.

Problems with complementarity constraints

Feasible region: $Ax + By + Cw \geq g$
$0 \leq y \perp w \geq 0$

Assume feasible region is bounded.

Then Balas result still holds:
can lift sequentially:

$$S^0 = \{(x,y,w) : Ax + By + Cw \geq g,\ y, w \geq 0\}$$

$$S^1 = \text{conv}\{(x,y,w) \in S^0 : y_1 = 0 \text{ or } w_1 = 0\}$$

$$S^2 = \text{conv}\{(x,y,w) \in S^1 : y_2 = 0 \text{ or } w_2 = 0\}$$

$$\vdots$$

$$S^n = \text{conv}\{(x,y,w) \in S^{n-1} : y_n = 0 \text{ or } w_n = 0\}$$

Then $S^n = \text{conv}\{(x,y,w) \in S^0 : y \perp w\}$

Lift and project procedure needs to change slightly:

Don't multiply by y_i and w_i and then linearize.

Instead, multiply by q_i and $1-q_i$,
with interpretation

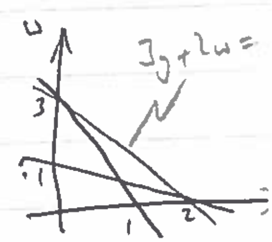
$$q_i = \begin{cases} \dfrac{y_i}{y_i+w_i} & \text{if } y_i+w_i>0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $y_i q_i = y_i$, $\quad w_i(1-q_i) = w_i$,
$y_i(1-q_i) = 0$, $\quad w_i q_i = 0$

Can show sequential convexification works with this
(Nguyen, Richard, Tawarmalani)
2011 IPCO proceedings)

- Use RLT framework,
  so multiply by all q_i, $1-q_i$
  and keep repeating.

Eg: min $y+w$
st. $3y+w \geq 3$, $y+2w \geq 2$, $0 \leq y \perp w \geq 0$
Soln to relaxation: $y=\frac{4}{5}$, $w=\frac{3}{5}$.



Cuts: $q(3y+w) \geq 3q \rightarrow 3y \geq 3q$ (not useful)

$(1-q)(3y+w) \geq 3(1-q) \rightarrow w \geq 3(1-q)$ ① $\quad$ $2①+3② \Rightarrow 3y+2w \geq 6$

$q(y+2w) \geq 2q \rightarrow y \geq 2q$ ②

$(1-q)(y+2w) \geq 2(1-q) \rightarrow 2w \geq 2(1-q)$ (not useful)