Lagrangian Relaxation

NgW II.3.6, pages 323-337.
Wolsey, Chapter 10.

Consider an IP

$$z_{IP} = \max c^T x$$

$$\text{s.t. } A^1 x \le b^1 \text{ (complicating constraints)} \quad (IP)$$
$$A^2 x \le b^2 \text{ (nice constraints)}$$
$$x \in \mathbb{Z}^n_+.$$

For any λ in $\mathbb{R}^{m_1}_+$, consider problem

$$LR(\lambda) \; z_{LR}(\lambda) = \max c^T x + \lambda^T(b^1 - A^1 x)$$
$$\text{s.t. } A^2 x \le b^2$$
$$x \in \mathbb{Z}^n_+$$

If $A^1 x \not\le b^1$, pay a "penalty" in objective function.

<u>Proposition</u> $LR(\lambda)$ is a relaxation of IP for all $\lambda \ge 0$, ie if x is feasible in IP then it is feasible in $LR(\lambda)$ with ~~larger~~ no smaller obj fn value.

<u>Proof</u> x clearly feasible in $LR(\lambda)$ if feasible in (IP).
Also, x feasible in (IP) $\Rightarrow$ $b^1 - A^1 x \ge 0 \Rightarrow \lambda^T(b^1 - A^1 x) \ge 0$ since $\lambda \ge 0$
$\Rightarrow c^T x + \lambda^T(b^1 - A^1 x) \ge c^T x$ $\boxed{//}$

Dual problem: $\min z_{LR}(\lambda)$ s.t. $\lambda \ge 0$ (LD)

<u>Proposition</u> Optimum value of (LD) is no smaller than optimum value of (IP)

Let $Q = \{x \in \mathbb{Z}^n_+ : A^2 x \leq b^2\}$.

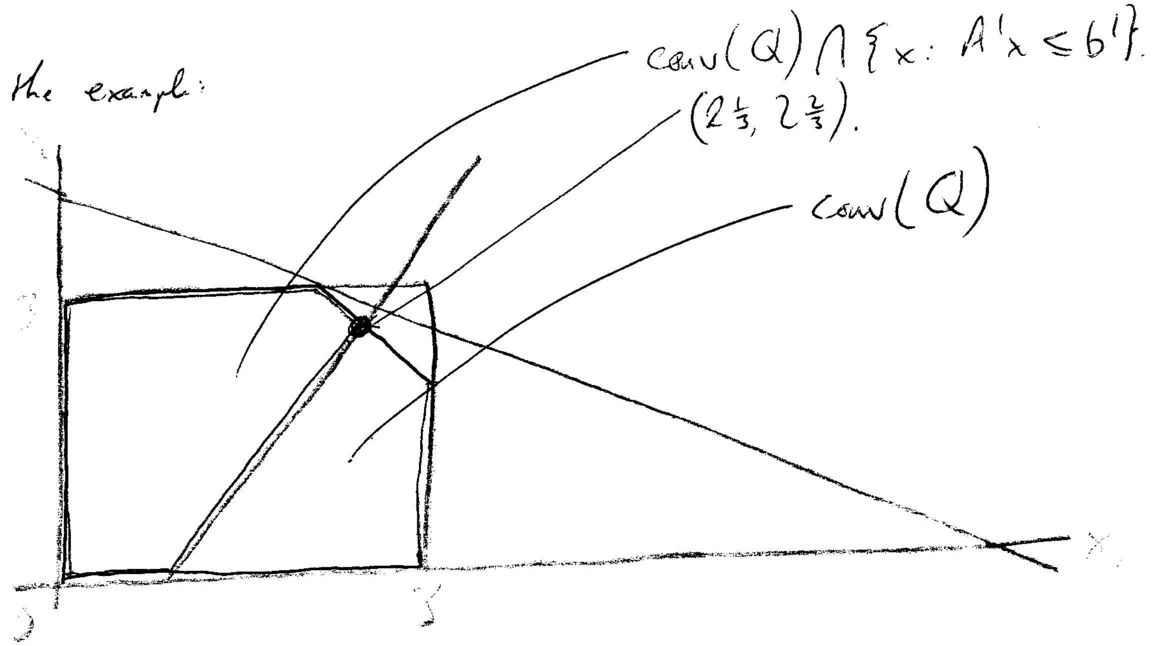
Q is the feasible region for $LR(\lambda)$.

~~Propos~~ **Theorem** The optimum value of the Lagrangian dual, z_{LD}, is given by

$$z_{LD} = \max\{c^T x : A^1 x \leq b^1, \ x \text{ in } \operatorname{conv}(Q)\}.$$

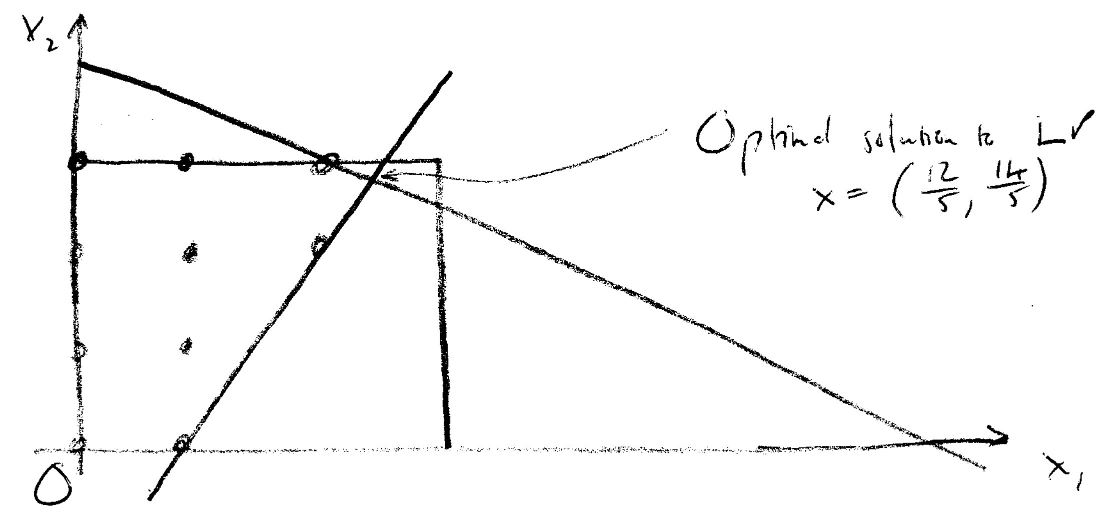
(Proof uses polyhedral theory to express $\operatorname{conv}(Q)$ in terms of its extreme points and its extreme rays. It then uses LP duality to get the result.)

Eg: In the example:



Example of Lagrangian Relaxation.

$$
\begin{aligned}
\max \quad & +x_1 \\
\text{s.t.} \quad & 2x_1 - x_2 \le 2 \quad \Big\} \; A^1 x \le b^1 \\
& x_1 + 2x_2 \le 8 \\
& x_1 \le 3 \\
& x_2 \le 3 \quad \Big\} \; A^2 x \le b^2. \\
& x_i \ge 0, \text{ integer.}
\end{aligned}
\qquad (IP)
$$



Optimal solution to LP relaxation, $x = \left(\frac{12}{5}, \frac{14}{5}\right)$

Lagrangian relaxation is:

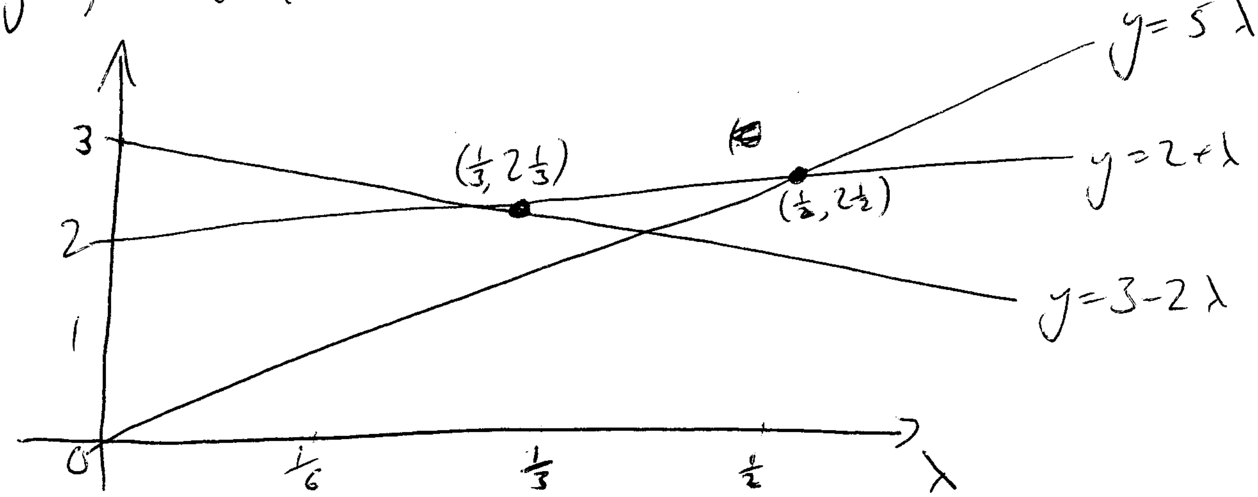
$$
\begin{aligned}
\max \quad & +x_1 + \lambda\big(2 - (2x_1 - x_2)\big) = 2\lambda + (1-2\lambda)x_1 + \lambda x_2 \\
\text{s.t.} \quad & x_1 + 2x_2 \le 8 \\
& x_1 \le 3 \\
& x_2 \le 3 \\
& x_i \ge 0, \ x_i \text{ integer.}
\end{aligned}
\qquad LR(\lambda)
$$

For each λ, soln is one of points $(0,0), (0,3), (2,3), (3,2), (3,0)$.

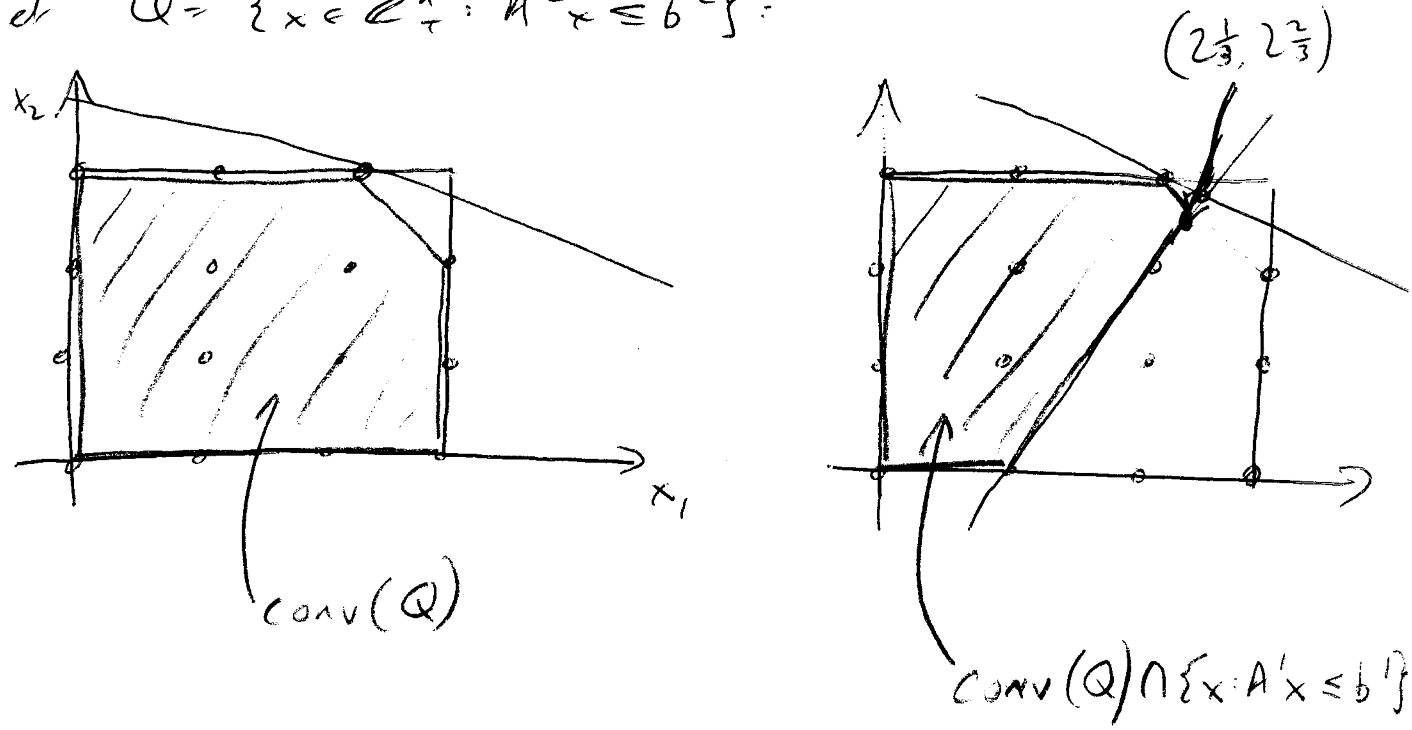
$$
\begin{aligned}
0 \le \lambda \le \tfrac{1}{3}: \quad & x^* = (3,2) \quad & z_{LR}(\lambda) = 3 - 2\lambda \\
\tfrac{1}{3} \le \lambda \le \tfrac{1}{2}: \quad & x^* = (2,3) \quad & z_{LR}(\lambda) = 2 + \lambda \\
\tfrac{1}{2} \le \lambda \quad & x^* = (0,3) \quad & z_{LR}(\lambda) = 5\lambda
\end{aligned}
$$

Lagrangian dual: $\min\limits_{\lambda \ge 0} z_{LR}(\lambda)$ Optimal: $\lambda = \frac{1}{3}$. Value $= 2\frac{1}{3}$.

Lagrangian dual:



Let $Q = \{x \in \mathbb{Z}^n_+ : A^2 x \leq b^2\}$.



$$Z_{LD} = \max \{c^T x : x \in \text{conv}(Q),\ A'x \leq b'\}$$

$$= c^T \begin{bmatrix} 2\frac{1}{3} \\ 2\frac{2}{3} \end{bmatrix} = 2\frac{1}{3} \quad \checkmark$$

How is the Lagrangian dual related to the LP relaxation?

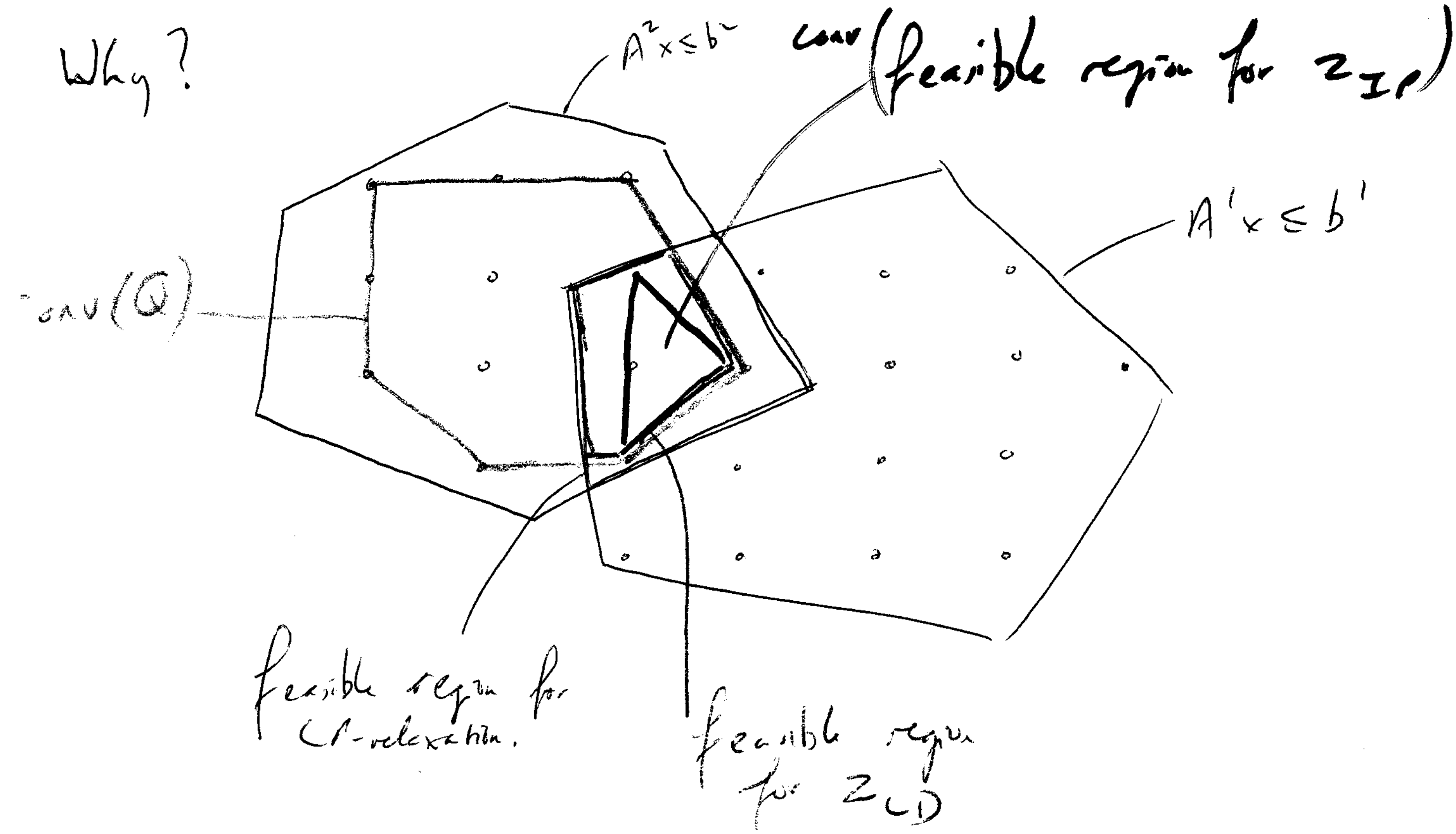
$$z_{IP} = \max\{c^Tx : A^1x \le b^1, A^2x \le b^2, x \text{ integer}\}.$$

$$z_{LD} = \max\{c^Tx : A^1x \le b^1, (x \text{ in } \operatorname{conv}\{x : A^2x \le b^2, x \text{ integer}\})\}.$$

$$z_{LP} = \max\{c^Tx : A^1x \le b^1, A^2x \le b^2, x \ge 0\}.$$

Theorem $z_{IP} \le z_{LD} \le z_{LP}$, and all of these inequalities can be strict.

Why?



Theorem $z_{LD} = z_{LP}$ for all c if all the extreme points of $\{x \text{ in } \mathbb{R}^n_+ : A^2x \le b^2\}$ are integral.

Why? Then $\operatorname{conv}\{x : A^2x \le b^2, x \in \mathbb{Z}^n_+\} = \{x \in \mathbb{R}^n_+ : A^2x \le b^2\}$.

So not very useful to keep only TU constraints and put others in objective.

Eg: Assignment ~~Knap~~ problem with budget constraint:

$$\max \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} x_{ij} = 1 \qquad j=1,\dots,n \qquad (1)$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad i=1,\dots,n \qquad (2)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij}x_{ij} \le b \qquad (3).$$

$$x \text{ binary}.$$

Several different Lagrangian relaxations available:

(i) Lagrangian relaxation with respect to (3):

$$\max \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} + \lambda\Big(b - \sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij}x_{ij}\Big)$$

$$\text{s.t.} \quad \sum_{i=1}^{n} x_{ij} = 1$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad LR_{(i)}(\lambda)$$

$$x \text{ binary}$$

Now, the feasible region here is the feasible region of an assignment problem, so ~~the~~ ~~stat per~~ $LR_{(i)}(\lambda)$ can be solved by solving the LP-relaxation and all the extreme points of $\{\cancel{x \in \mathbb{Z}\ \text{binary}}\ 0 \le x \le 1 : A^2 x \le b^2\}$ are integer. $\therefore Z_{LD} = Z_{LP}$, so the Lagrangian relaxation gains nothing over the LP-relaxation.

(ii) Lagrangian relaxation wrt (1) and (2):

$$\max \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{i=1}^{n} u_i\Big(1-\sum_{j=1}^{n}x_{ij}\Big) + \sum_{j=1}^{n} v_j\Big(1-\sum_{i=1}^{n}x_{ij}\Big) \qquad LR_{(ii)}(u,v)$$

$$\text{st.} \quad \sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij}x_{ij} \le b$$

$$x \text{ binary}.$$

Lagrangian relaxation is a Knapsack problem, so we ~~be~~ now get a better bound on z_{IP}.

Drawback: Harder to solve $LR_{(ii)}(u,v)$ than $LR_{(i)}(\lambda)$.

Other relaxations possible.

Tradeoff: ease of solving relaxation vs. strength of bound.

How do we find z_{LD}? (See pages LR10 & LR11)

Assuming we can find $z_{LR}(\lambda)$ easily enough,

we change λ and resolve:

If ith constraint of $A^1 x \le b^1$ is violated, increase corresponding λ

If ith constraint of $A^1 x \le b^1$ has a large positive slack, decrease corresponding λ.

Eg 1-tree relaxation of TSP: (Held-Karp).

Write TSP as:

$$\min \sum c_e x_e$$
$$\text{s.t.} \quad \sum_{e \in \delta(v)} x_e = 2$$
$$x_e \text{ is connected}$$
$$x_e \text{ binary}$$

or

$$\min \sum c_e x_e + \sum_{v=2}^{|V|} \lambda_v \Big(\sum_{e \in \delta(v)} x_e - 2 \Big)$$
$$\text{s.t.} \quad \sum_{e \in \delta(1)} x_e = 2$$
$$x \text{ is a spanning tree on vertices } 2, \ldots, |V|.$$
$$x \text{ binary.}$$

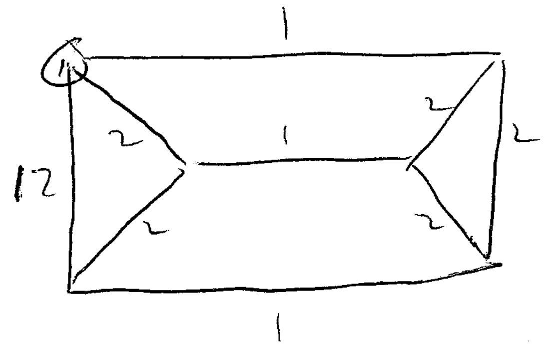
Here, λ can be positive or negative, because we relax an equality constraint.

The feasible region here is a 1-tree:

ie: a spanning tree on vertices $2, \ldots, n$ + 2 extra edges, connecting 1 to rest of graph.

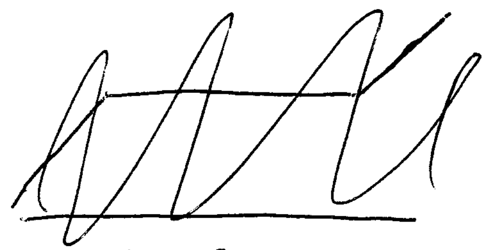
Can solve this by finding a minimum weight spanning tree, and adding the cheapest extra edge.

Eg:

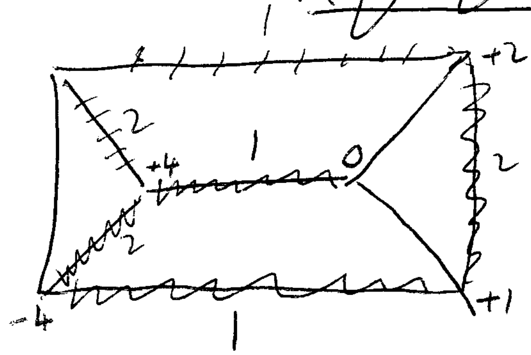


Other edges have length 10.

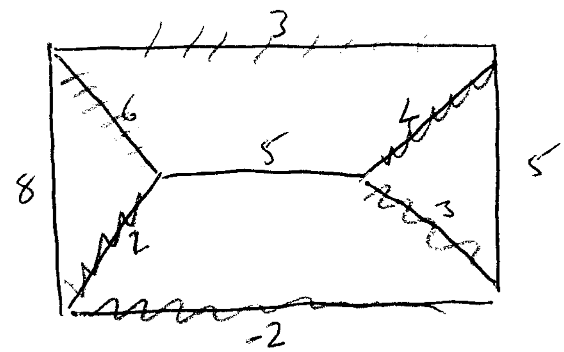
With $\lambda = 0$:
Spanning tree:



Note: if increment by degree -2, then solve it in 2 iterations



Change λ as shown:



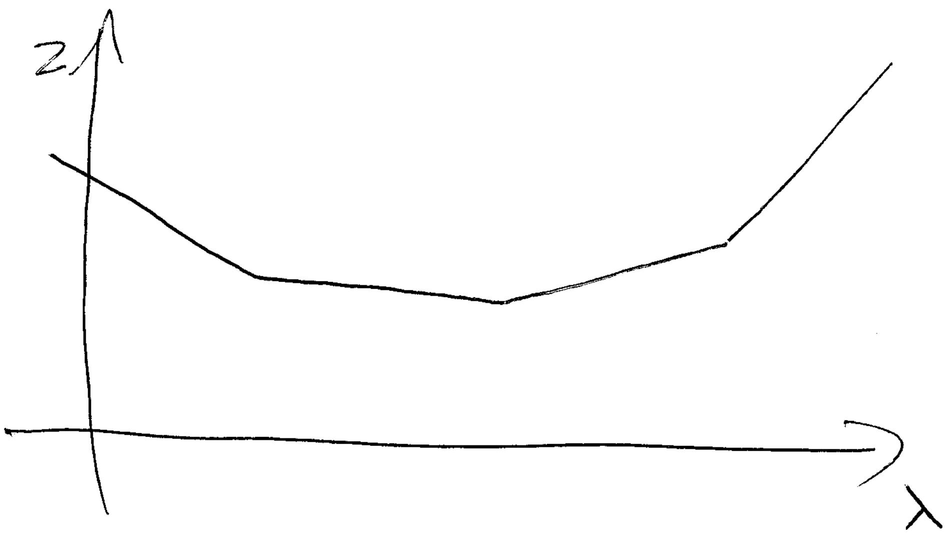
Solves TSP.

Note: around the ~~the~~ tour, total change due to λ's is $2\Sigma\lambda_i$, so all tours changed by same amount.

Solving Lagrangian Relaxations:

Lagrangian dual:

$$\begin{aligned} \min \quad & z_{LR}(\lambda) \\ \text{s.t.} \quad & \cancel{z \geq z_{LR}(\lambda)} \quad \cancel{\forall \lambda} \\ & \lambda \geq 0 \end{aligned}$$



Can show that $z_{LR}(\lambda)$ is a convex function.

Gradually build up an approximation to this function.

$$\begin{aligned} z_{LR}(\lambda) = \max \quad & c^T x + \lambda^T (b' - A' x) \\ \text{s.t.} \quad & A^2 x \leq b^2 \\ & x \in \mathbb{Z}^n_+ . \end{aligned}$$

Solved by x^λ

Value $c^T x^\lambda + \lambda^T (b' - A' x^\lambda)$

So build up as:

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & z \geq c^T x^\lambda + \overbrace{(b' - A' x^\lambda)}^{\text{subgradient.}}{}^T \lambda \\ & \lambda \geq 0 \end{aligned}$$

Piecewise-linear approximation

So, the example on page LR3:

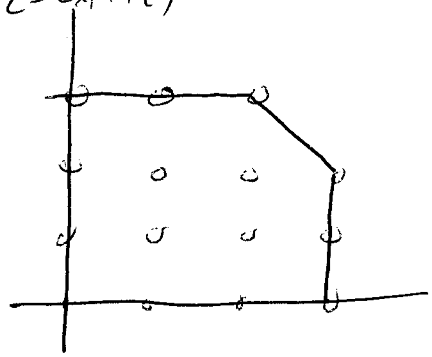
$$z_{LR}(\lambda) = \max \quad 2\lambda + (1-2\lambda)x_1 + \lambda x_2 = x_1 + \lambda(2 - 2x_1 + x_2)$$

s.t. $\quad x_1 + 2x_2 \leq 8$

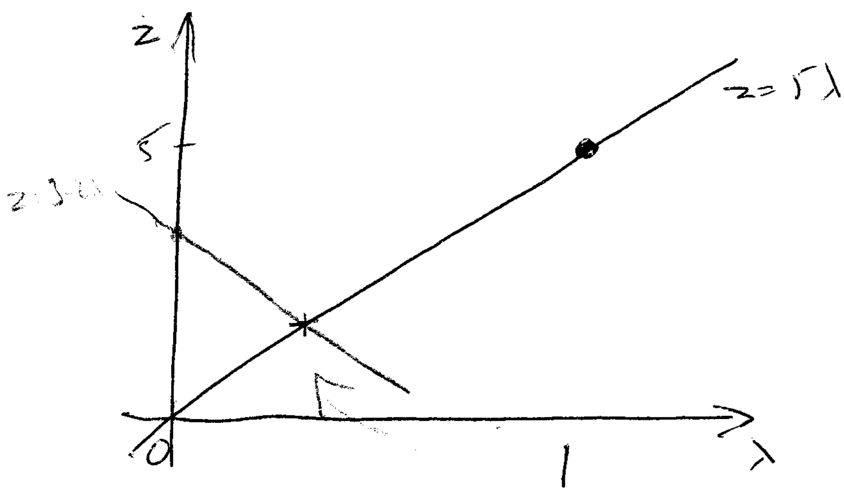
$\quad x_1 \leq 3$

$\quad x_2 \leq 3$

$\quad x_i \geq 0$, x_i integer.



Try $\lambda = 1$, say: So max $2 - x_1 + x_2$ over these 15 points.
Get $x = (0,3)$, $z = 5$, slope $= 5$ $(= 2 - 2x_1 + x_2)$



min z
s.t. $z \geq 5\lambda$
$\lambda \geq 0$

Soln: $\lambda = 0$.
Then gives ~~z_{LR}~~ max x_1, so $(3,2)$ eg.
So $z = 3$, slope $= -2$
or: $(3,0)$: $z = 3$, slope $= -4$

min z
s.t. $z \geq 5\lambda$
$z \geq 3 - 2\lambda$
$\lambda \geq 0$

Minimized where these lines cross:
$5\lambda = 3 - 2\lambda \Rightarrow \lambda = 3/7$, cross at $z = \frac{15}{7}$
$\Rightarrow$ so max $\frac{6}{7} + \frac{1}{7}x_1 + \frac{3}{7}x_2$
$\Rightarrow x = (2,3)$, then $z = \frac{17}{7}$.
slope $= 1$ $\quad z \geq 2 + \lambda$

Add this constraint [illegible]