

Lagrangian RelaxationN&W II.3.6, pages 323-337.
Wolsey, Chapter 10.

Consider an IP

$$z_{IP} = \max c^T x$$

$$\text{s.t. } \begin{aligned} A^1 x &\leq b^1 && \text{(coupling constraints)} \\ A^2 x &\leq b^2 && \text{(nice constraints)} \end{aligned} \quad (IP)$$

$$x \in \mathbb{Z}_+^n.$$

For any $\lambda \in \mathbb{R}_+^{m_1}$, consider problem

$$LR(\lambda) z_{LR}(\lambda) = \max c^T x + \lambda^T (b^1 - A^1 x)$$

$$\text{s.t. } \begin{aligned} A^2 x &\leq b^2 \\ x &\in \mathbb{Z}_+^n \end{aligned}$$

If $A^1 x \not\leq b^1$, pay a "penalty" in objective function.

Proposition $LR(\lambda)$ is a relaxation of IP for all $\lambda \geq 0$, i.e. if x is feasible in IP then it is feasible in $LR(\lambda)$ with ~~no~~ no smaller obj. value.

Proof x clearly feasible in $LR(\lambda)$ if feasible in (IP).

$$\text{Also, } x \text{ feasible in (IP)} \Rightarrow b^1 - A^1 x \geq 0 \Rightarrow \lambda^T (b^1 - A^1 x) \geq 0$$

$$\Rightarrow c^T x + \lambda^T (b^1 - A^1 x) \geq c^T x$$



$$\text{Dual problem: } \min z_{LR}(\lambda) \quad (LD)$$

$$\text{s.t. } \lambda \geq 0$$

Proposition Optimum value of (LD) is no smaller than optimum value of (IP)

Let $Q = \{x \in \mathbb{Z}_+^n : A^2 x \leq b^2\}$.

Q is the feasible region for $LR(\lambda)$.

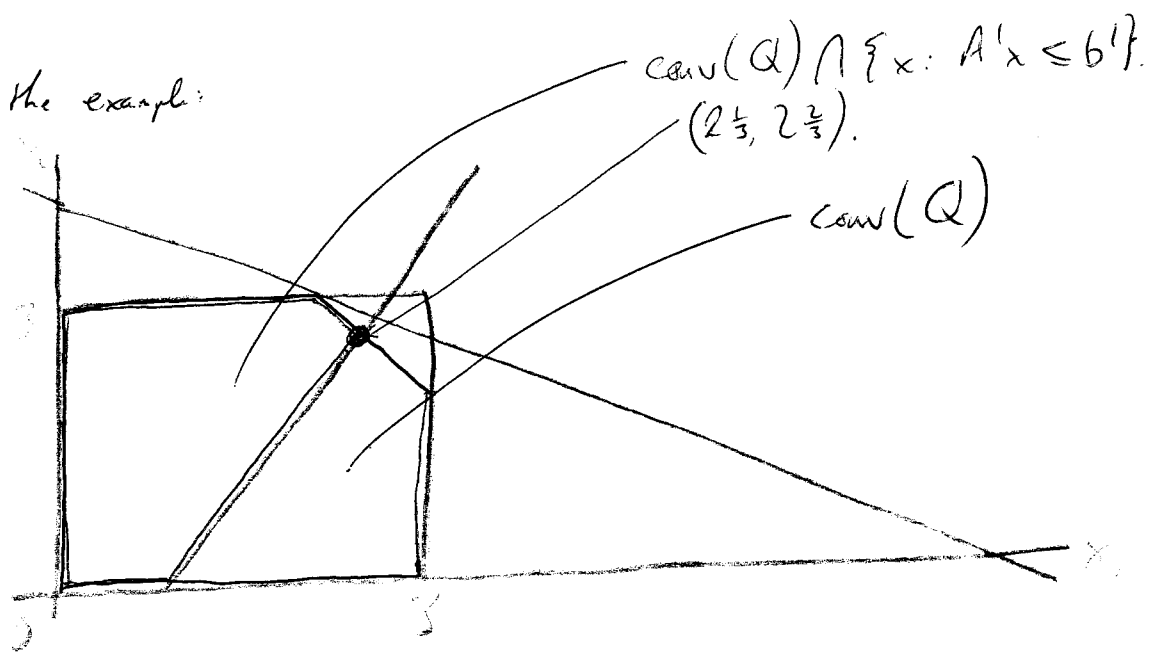
Proposition
Theorem

The optimum value of the Lagrangian dual, z_{LD} , is given by

$$z_{LD} = \max \{c^T x : A^1 x \leq b^1, x \in \text{conv}(Q)\}$$

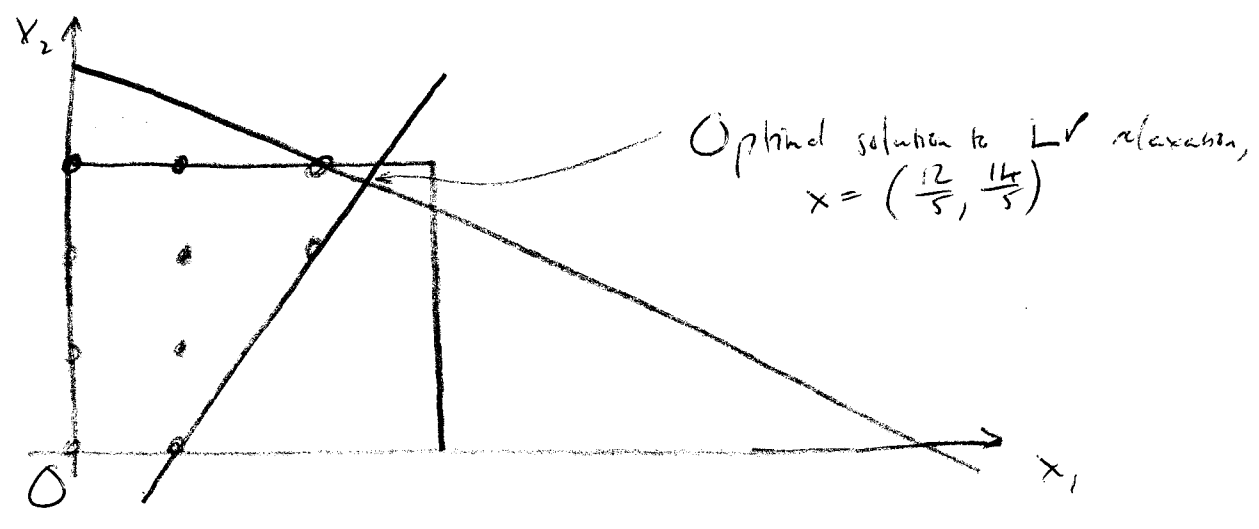
(Proof uses polyhedral theory to express $\text{conv}(Q)$ in terms of its extreme points and its extreme rays. It then uses LP duality to get the result.)

Eg: In the example:



Example of Lagrangian Relaxation.

$$\begin{aligned}
 \max \quad & +x_1 \\
 \text{s.t.} \quad & 2x_1 - x_2 \leq 2 \quad \left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\} A^1 x \leq b^1 \\
 & x_1 + 2x_2 \leq 8 \quad \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} A^2 x \leq b^2 \\
 & x_1 \leq 3 \\
 & x_2 \leq 3 \\
 & x_i \geq 0, \text{ integer.}
 \end{aligned} \tag{IP}$$



Lagrangian relaxation is:

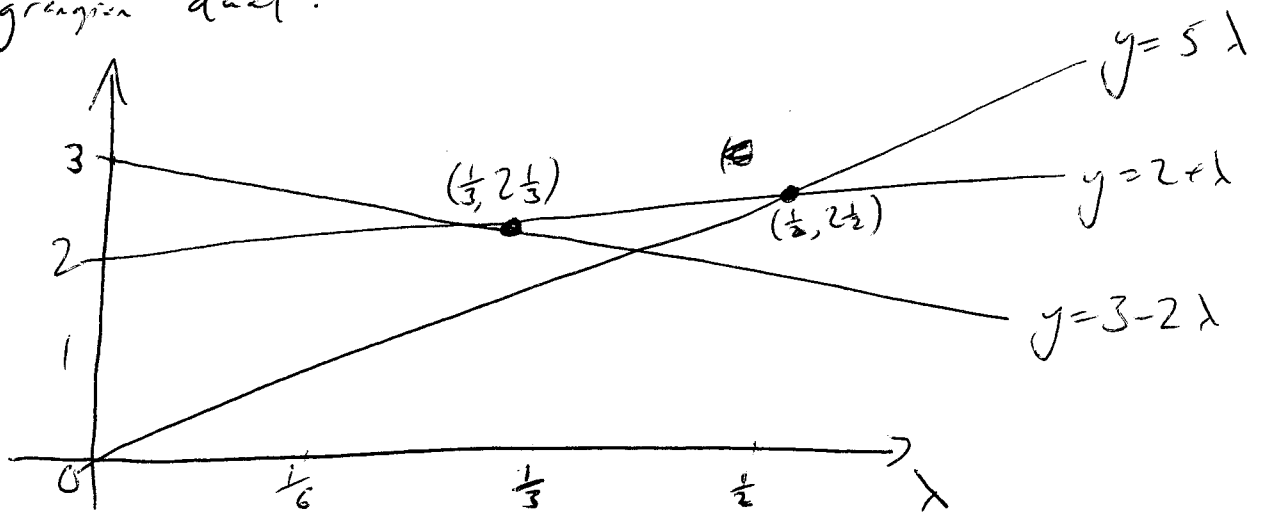
$$\begin{aligned}
 \max \quad & +x_1 + \lambda (2 - (2x_1 - x_2)) = 2\lambda + (1-2\lambda)x_1 + \lambda x_2 \\
 \text{s.t.} \quad & x_1 + 2x_2 \leq 8 \quad \text{LR}(\lambda) \\
 & x_1 \leq 3 \\
 & x_2 \leq 3 \\
 & x_i \geq 0, x_i \text{ integer.}
 \end{aligned}$$

For each λ , soln is one of points $(0,0), (0,3), (2,3), (3,2), (3,0)$.

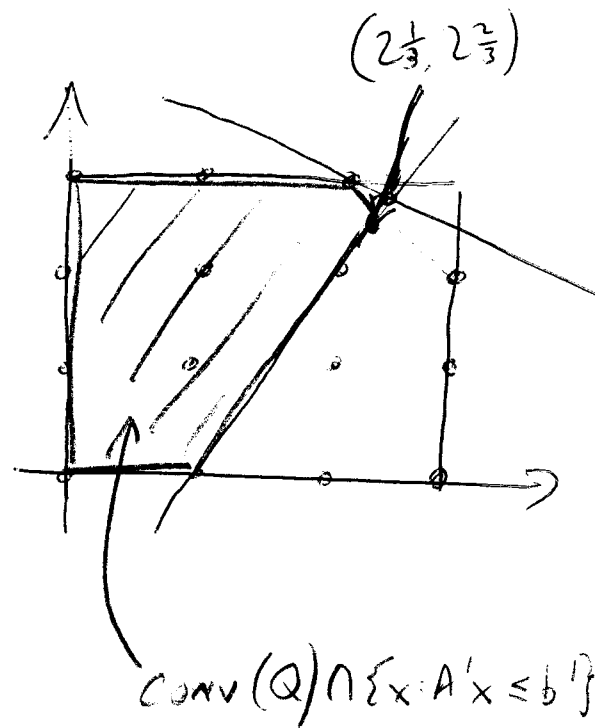
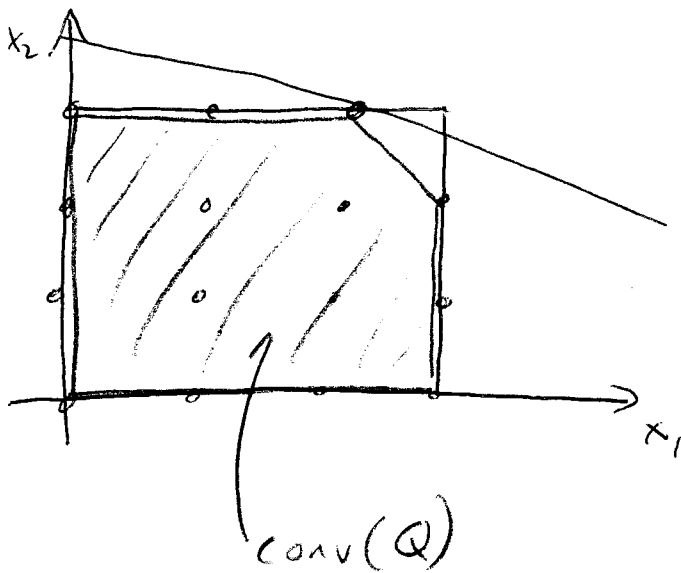
$0 \leq \lambda \leq \frac{1}{3}$	$x^* = (3, 2)$	$z_{LR}(\lambda) = 3 - 2\lambda$
$\frac{1}{3} \leq \lambda \leq \frac{1}{2}$	$x^* = (2, 3)$	$z_{LR}(\lambda) = 2 + \lambda$
$\frac{1}{2} \leq \lambda$	$x^* = (0, 3)$	$z_{LR}(\lambda) = 5\lambda$

Lagrangian dual: $\min_{\lambda \geq 0} z_{LR}(\lambda)$ Optimal: $\lambda = \frac{1}{3}$. Value = $2\frac{1}{3}$.

Lagrangean dual:



Let $Q = \{x \in \mathbb{Z}_+^n : A^2 x \leq b^2\}$:



$$\begin{aligned}
 z_{LD} &= \max \{c^T x : x \in \text{conv}(Q), A'x \leq b'\} \\
 &= c^T \begin{bmatrix} 2\frac{1}{3} \\ 2\frac{2}{3} \end{bmatrix} = 2\frac{1}{3} \quad \checkmark
 \end{aligned}$$

How is the Lagrangian dual related to the LP relaxation?

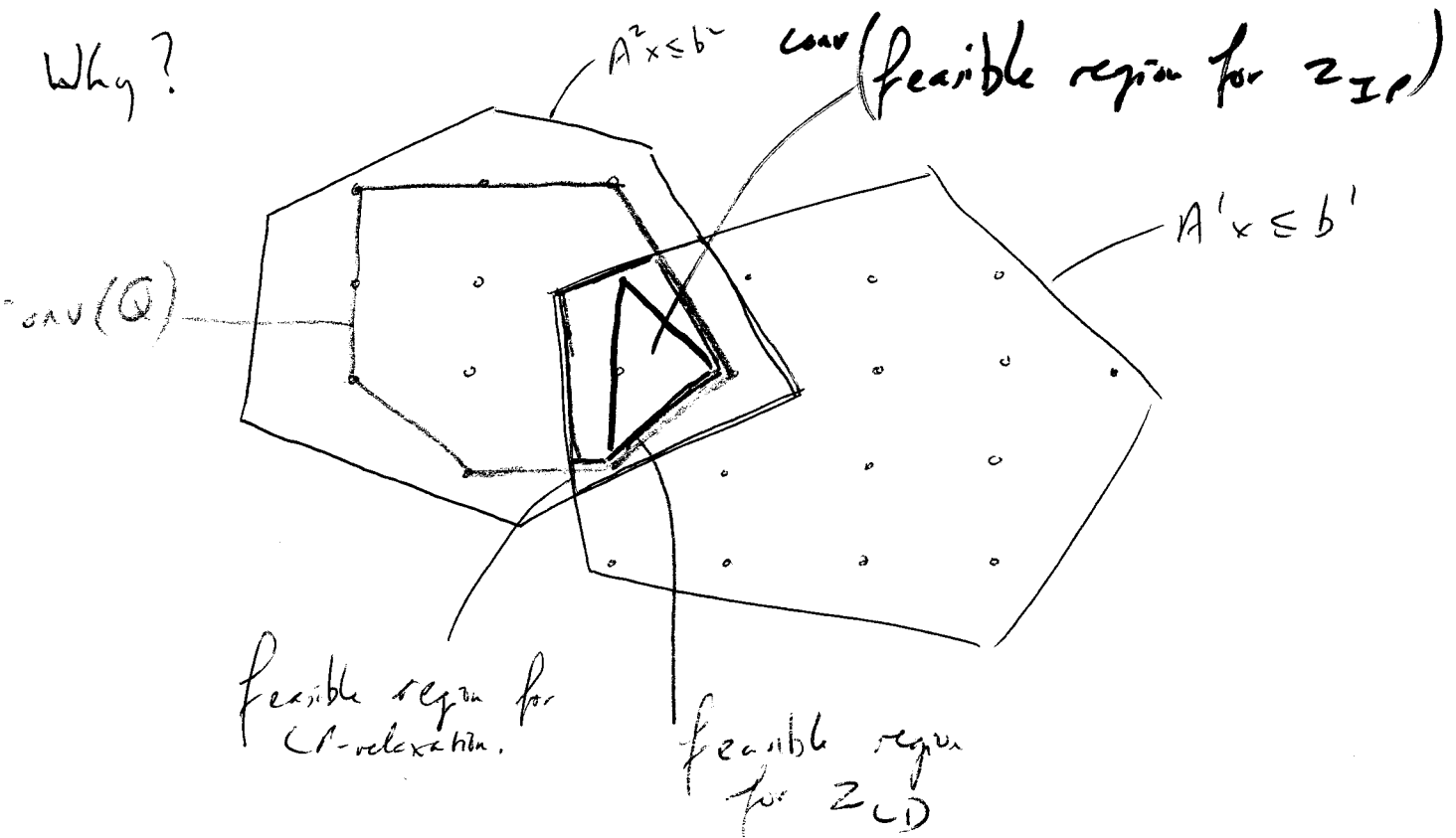
$$z_{IP} = \max \{ c^T x : A^1 x \leq b^1, A^2 x \leq b^2, x \text{ integer} \}$$

$$z_{LD} = \max \{ c^T x : A^1 x \leq b^1, (x \text{ in conv} \{ x : A^2 x \leq b^2, x \text{ integer} \}) \}$$

$$z_{LP} = \max \{ c^T x : A^1 x \leq b^1, A^2 x \leq b^2, x \geq 0 \}$$

Theorem $z_{IP} \leq z_{LD} \leq z_{LP}$, and all of these inequalities can be strict.

Why?



Theorem $z_{LD} = z_{LP}$ for all c if all the extreme points of $\{ x \in \mathbb{R}_+^n : A^2 x \leq b^2 \}$ are integral.

Why? Then $\text{conv} \{ x : A^2 x \leq b^2, x \in \mathbb{Z}_+^n \} = \{ x \in \mathbb{R}_+^n : A^2 x \leq b^2 \}$.

So not very careful to keep only TU constraints and put other in objective.

Eg: ~~Flow~~ Assignment problem with budget constraint:

$$\begin{aligned} \max \quad & \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{i=1}^n x_{ij} = 1 \quad j = 1, \dots, n \quad (1) \\ & \sum_{j=1}^n x_{ij} = 1 \quad i = 1, \dots, n \quad (2) \\ & \sum_{i=1}^n \sum_{j=1}^n k_{ij} x_{ij} \leq b \quad (3) \\ & x \text{ binary.} \end{aligned}$$

Several different Lagrangian relaxations available:

(i) Lagrangian relaxation with respect to (3):

$$\begin{aligned} \max \quad & \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} + \lambda (b - \sum_{i=1}^n \sum_{j=1}^n k_{ij} x_{ij}) \\ \text{s.t.} \quad & \sum_{i=1}^n x_{ij} = 1 \\ & \sum_{j=1}^n x_{ij} = 1 \\ & x \text{ binary} \end{aligned} \quad LR_{(i)}(\lambda)$$

Now, the feasible region here is the feasible region of an assignment problem, so the ~~subproblem~~ $LR_{(i)}(\lambda)$ can be solved by solving the LP-relaxation, and all the extreme points of $\{x \in \mathbb{Z}^{n^2} : A^2 x \leq b^2\}$ are integer. $\therefore Z_{LP} = Z_{LP}$, so the Lagrangian relaxation gains nothing over the LP-relaxation.

(ii) Lagrangian relaxation w.r.t (1) and (2):

$$\begin{aligned} \max \quad & \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} + \sum_{i=1}^n u_i (1 - \sum_{j=1}^n x_{ij}) + \sum_{j=1}^n v_j (1 - \sum_{i=1}^n x_{ij}) \\ \text{s.t.} \quad & \sum_{i=1}^n \sum_{j=1}^n k_{ij} x_{ij} \leq b \\ & x \text{ binary.} \end{aligned} \quad LR_{(ii)}(u, v)$$

Lagrange relaxation is a Knapsack problem, so we ~~can~~ now get a better bound on z_{IP} .

Drawback: Harder to solve $LR_{(ii)}(u, v)$ than $LR_{(i)}(\lambda)$.

Other relaxations possible.

Tradeoff: ease of solving relaxation vs. strength of bound.

How do we find z_{LD} ? (See page LR10 & LR11)

Assuming we can find $z_{LR}(\lambda)$ easily enough,

we change λ and resolve:

If the constraint of $A^d x \leq b^d$ is violated, increase corresponding λ .

If the constraint of $A^s x \leq b^s$ has a large positive slack, decrease corresponding λ .

Ex. 1-tree relaxation of TSP: (Held-Karp)

Write TSP as:

$$\min \sum c_e x_e$$

$$\text{s.t. } \sum_{e \in \delta(v)} x_e = 2$$

x_e is connected

x_e binary

$$\text{or } \min \sum c_e x_e + \sum_{v \in V} \lambda_v (\sum_{e \in \delta(v)} x_e - 2)$$

$$\text{s.t. } \sum_{e \in \delta(1)} x_e = 2$$

~~x is connected~~ x is a spanning tree on vertices $2, \dots, |V|$.
 x binary.

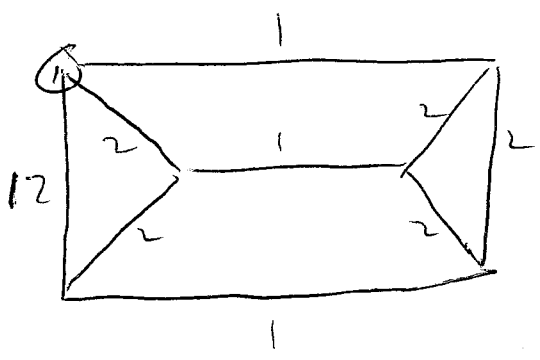
Here, λ can be positive or negative, because we relax an equality constraint.

The feasible region here is a 1-tree:

ie: a spanning tree + 2 extra edges,
 on vertex $2, \dots, n$ connecting 1 to rest of graph.

Can solve this by finding a minimum weight spanning tree, and adding the cheapest extra edge.

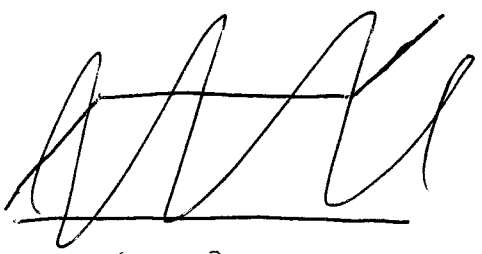
Eg:



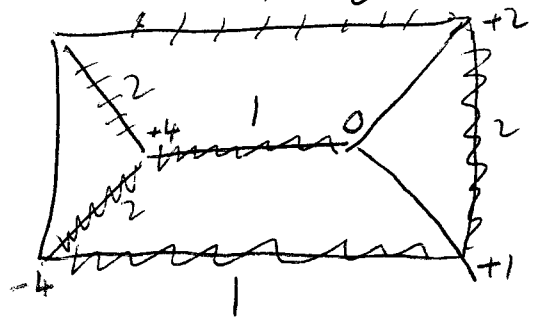
Other edges have length 10.

With $\lambda = 0$:

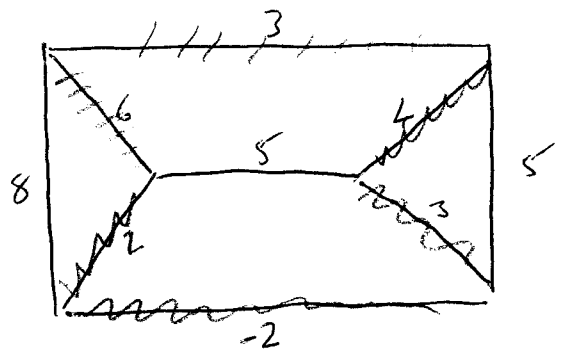
Spanning tree:



Note if increment by degree-2, then solve it in 2 iterations



Change λ as shown:



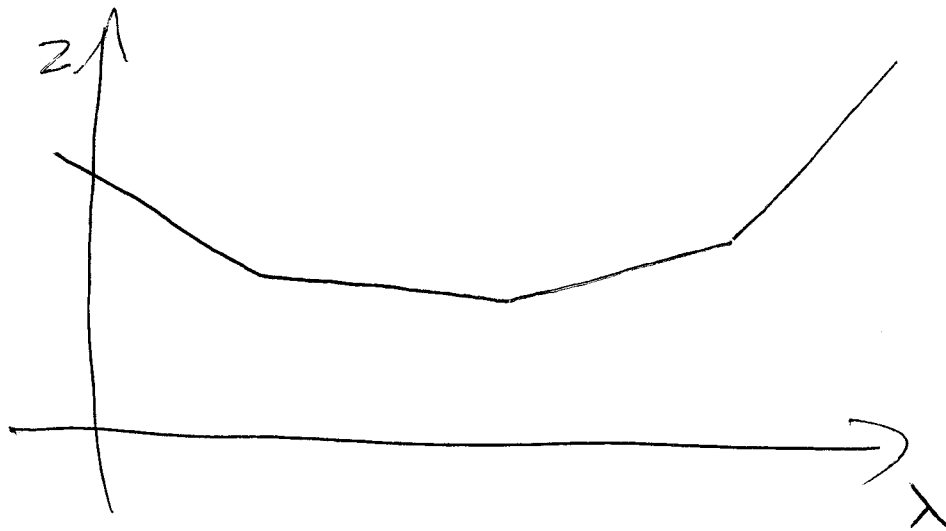
Solves TSP.

Note: around the tour, total change due to λ_i is $2 \sum \lambda_i$, so all tours changed by same amount.

Solving Lagrangian Relaxation:

Lagrangian dual:

$$\begin{aligned} \min \quad & z_{LR}(\lambda) \\ \text{s.t.} \quad & \cancel{z_{LR}(\lambda)} \quad \cancel{\lambda} \\ & \lambda \geq 0 \end{aligned}$$



Can show that $z_{LR}(\lambda)$ is a convex function.

Gradually build up an approximation to this function.

$$\begin{aligned} z_{LR}(\lambda) = \max \quad & c^T x + \lambda^T (b - A^T x) \\ \text{s.t.} \quad & Ax \leq b \\ & x \in \mathbb{Z}_+^n \end{aligned}$$

$$\left(\begin{matrix} \lambda^T \\ \lambda^T \end{matrix} \right)^T \begin{pmatrix} x \\ x \end{pmatrix} = \lambda^T (x - x)$$

Solved by ~~x^*~~ x^*
 Value: $c^T x^* + \lambda^T (b - A^T x^*)$

subgradient,

So build up as:

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & z \geq c^T x^* + (b - A^T x^*)^T \lambda \\ & \lambda \geq 0 \end{aligned}$$

Piecewise-linear approximation

So, an example on page LR3:

$$z_{LP}(\lambda) = \max_x \quad 2\lambda + (1-2\lambda)x_1 + \lambda x_2 = x_1 + \lambda(2-2x_1+x_2)$$

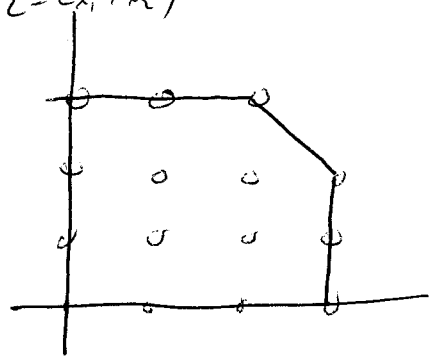
s.t.

$$x_1 + 2x_2 \leq 8$$

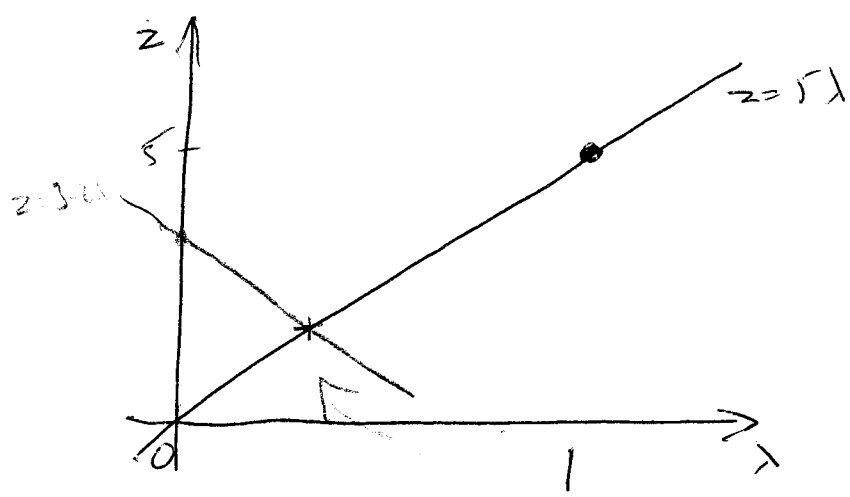
$$x_1 \leq 3$$

$$x_2 \leq 3$$

$$x_i \geq 0, x_i \text{ integer.}$$



Try $\lambda = 1$, say: So max $2 - x_1 + x_2$ over these 15 points.
 Get $x = (0, 3)$, $z = 5$, slope = $\frac{5}{(2-2x_1+x_2)}$



min z

s.t. $z \geq 5\lambda$

$\lambda \geq 0$

Solve: $\lambda = 0$.
 Then gives $z = \max x_1$, so $(3, 2)$ eg.
 So: $z = 3$, slope = -2
 or: $(3, 0)$: $z = 3$, slope = -4

min z

s.t. $z \geq 5\lambda$

$z \geq 3 - 2\lambda$

$\lambda \geq 0$

Minimized where these lines cross:
 $5\lambda = 3 - 2\lambda \Rightarrow \lambda = \frac{3}{7}$, cross at $z = \frac{15}{7}$
 \Rightarrow so max $\frac{6}{7} + \frac{1}{7}x_1 + \frac{3}{7}x_2$
 $\Rightarrow x = (2, 3)$, then $z = \frac{17}{7}$.
 slope = 1 , $z \geq 3 + \lambda$
 Add this constraint unless...