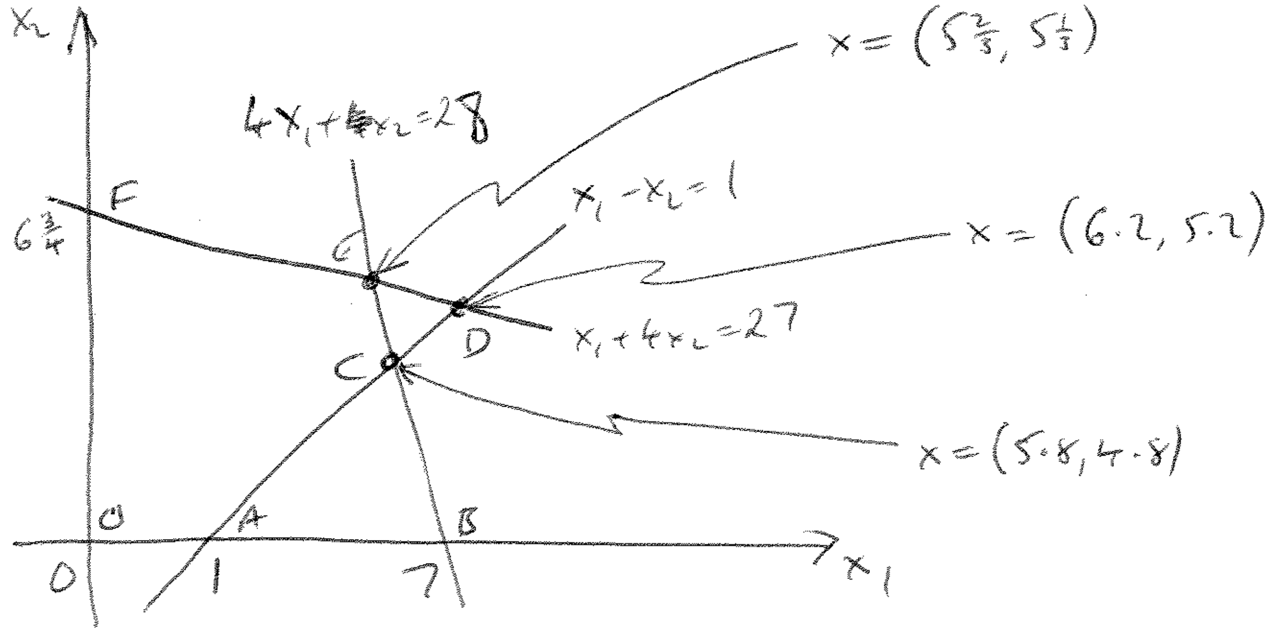
NRW, page 346, Q14.



Now, $z_{LD} = \max\{c^T x : A^1 x \le b^1,\ x \in \text{conv}(Q)\}$

where $Q = \{x \in \mathbb{Z}^n_+ : A^2 x \le b^2\}$.

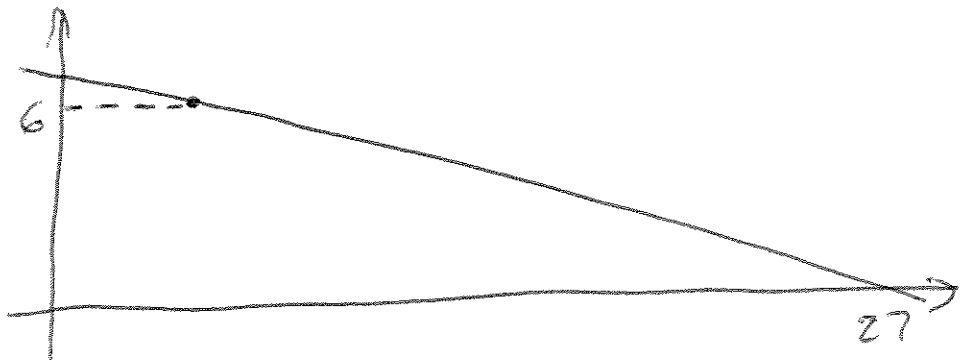
Optimal solution to LP: point G, $x = (5\frac{2}{3}, 5\frac{1}{3})$, $z_{LP} = 38$.

(i) (a) Dualize first two constraints:
Then $\text{conv}(Q) = \{x \in \mathbb{R}^2_+ : x_1 - x_2 \le 1\}$, so $z_{LD} = z_{LP}$

(b) Dualize last two constraints:
Then $\text{conv}(Q) = \{x \in \mathbb{R}^2_+ : 4x_1 + x_2 \le 28\}$, so $z_{LD} = z_{LP}$

(c) Dualize first and last constraints.
Then $\text{conv}(Q) = \text{conv}(\{x \in \mathbb{Z}^2_+ : x_1 + 4x_2 \le 27\})$
$= \text{conv}\{(0,0), (27,0), (0,6), (3,6)\} \ni (5\frac{2}{3}, 5\frac{1}{3})$.
So $z_{LD} = z_{LP}$.

(ii) Dualize first and third constraints:



Choose objective so that optimal soln to LP relaxation is $(0, 6\frac{3}{4})$

Eg: $\max \; x_2$.
Then $z_{LP} = 6\frac{3}{4}$, but $z_{LD} = 6$.

(iii) (a) Dualize $4x_1 + x_2 \leq 28$:
Then $Q = \{x \in \mathbb{Z}^2_+ : x_1 - x_2 \leq 1,\ x_1 + 4x_2 \leq 27\}$
$= \{x \in \mathbb{Z}^2_+ : x \in \text{polygon}(OADF)\}$
so $\text{conv}(Q) = \text{conv}\{(0,0), (1,0), (6,5), (3,6), (0,6)\}$

The line between $(6,5)$ and $(3,6)$ crosses $4x_1 + x_2 = 28$ at $(5\frac{8}{11}, 5\frac{1}{11})$,
so $z_{LD} = 36\frac{6}{11}$.

(b) Delete $x_1 + 4x_2 \leq 27$:
Then $Q = \{x \in \mathbb{Z}^2_+ : x \in \text{polygon}(OAC\,(0,28))\}$
so $\text{conv}(Q) = \text{conv}\{(0,0), (1,0), (5,4), (5,8), (0,28)\}$

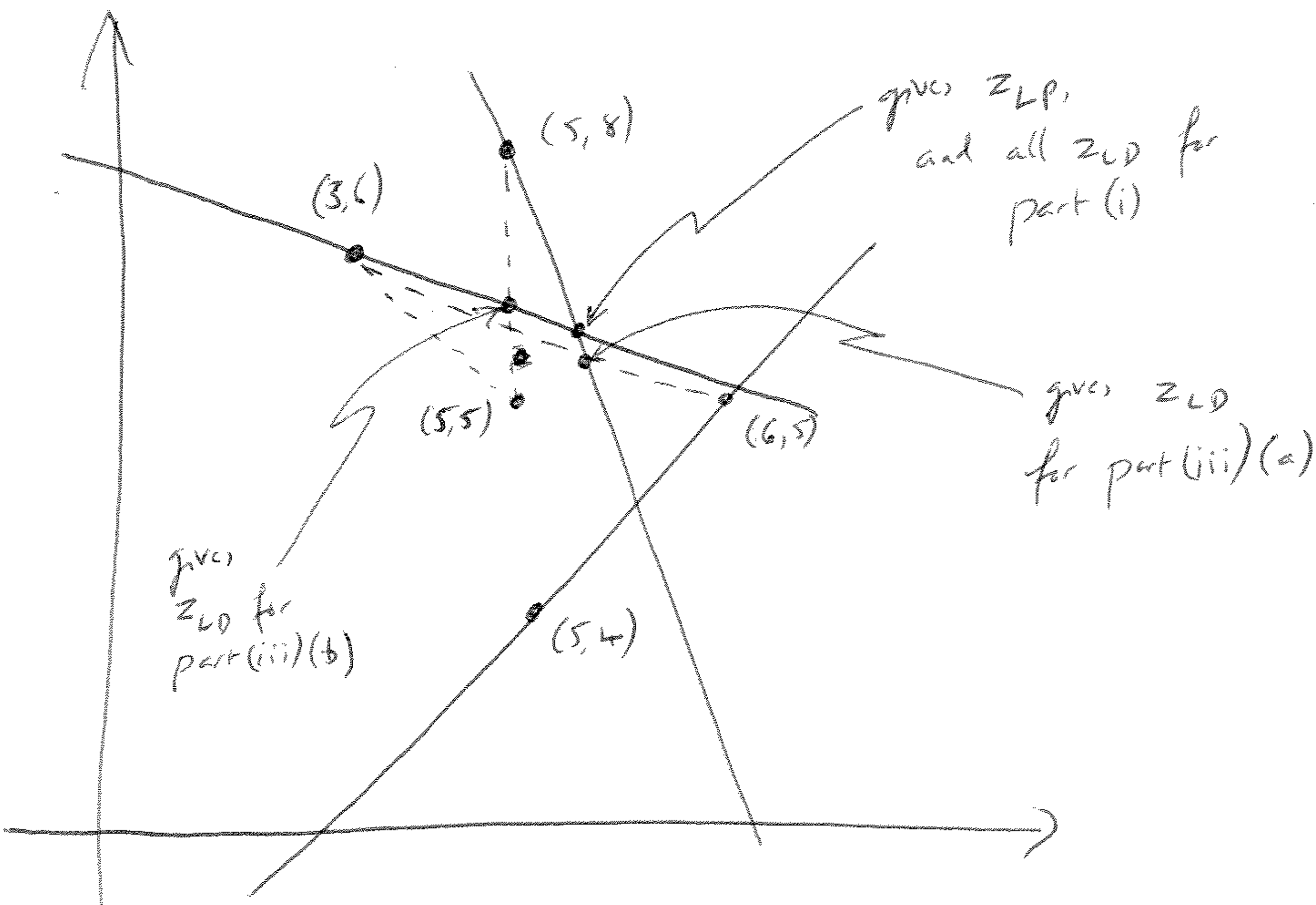
The line between $(5,4)$ and $(5,8)$ crosses $x_1 + 4x_2 = 27$ at $(5, 5.5)$
so $z_{LD} = 37\frac{1}{2}$

(c) Dualize $x_1 - x_2 \leq 1$:

Then $Q = \{x \in \mathbb{Z}^2_+ : x \text{ in polygon } (OBEF)\}$
so $conv(Q) = conv\{(0,0), (7,0), (6,4), (5,5), (3,6), (0,6)\}$

The point $(3,6)$ is the best point in $conv(Q)$ and satisfies $x_1 - x_2 \leq 1$.
So $z_{LD} = 36$.

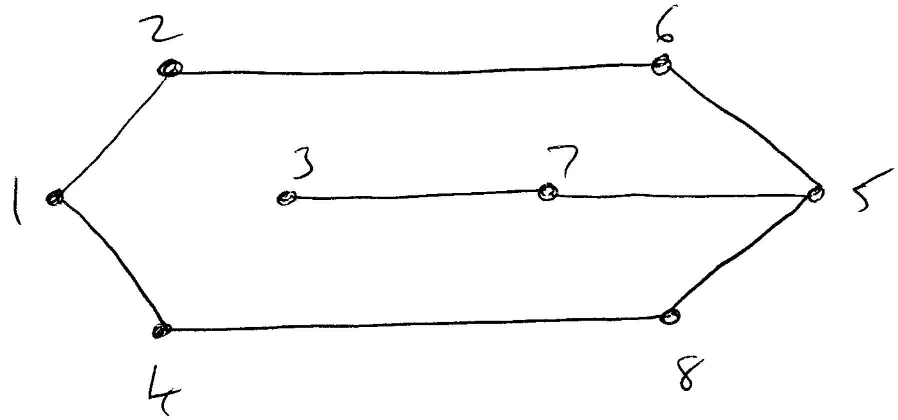
(v)(e)



Extreme integer points are $(0,0), (0,6), (3,6), (5,5), (5,4), (1,0)$.

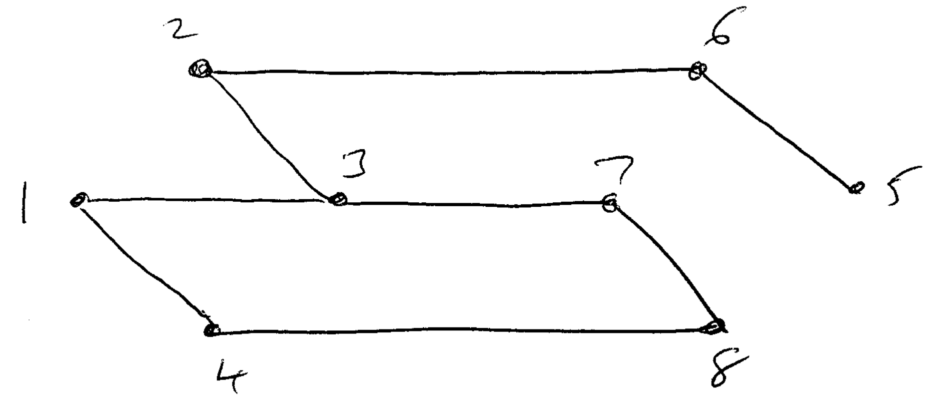
2. Two possible solutions to 1-tree relaxation with $\lambda = 0$:

(a)



Subgradient: $(0, 0, \underset{\substack{\uparrow \\ \text{2-degree}}}{1}, 0, -1, 0, 0, 0) =: \xi^1$

(b)



Subgradient: $(0, 0, -1, 0, 1, 0, 0, 0) =: \xi^2$

Average of these subgradients is $\xi = \frac{1}{2}\xi^1 + \frac{1}{2}\xi^2 = 0$.

Since $0 \in \text{conv}\{\text{set of subgradients}\}$, this $\lambda = 0$ is optimal.

3. Let x_{ij}^{k} = flow of commodity k on arc (i,j).

$\lambda = 0$: Optimal soln is $x_{ba}^{A} = x_{ag}^{A} = x_{gc}^{A} = 1$,

$$x_{fd}^{B} = x_{dc}^{B} = x_{ca}^{B} = x_{ag}^{B} = x_{gc}^{B} = 1,$$

all other $x_{ij}^{k} = 0$.

Value: $7+2+1+3+4+3+2 = 22$.

If only dualize the upper bound constraints, the resulting IP is separable into two single commodity network flow problems. For these problems, the constraint matrix is TU.

Hence, optimal value of Lagrangian relaxation
= optimal value of LP relaxation.

4. **Iteration 1** : $z = 28, \quad y_1 = y_2 = 0.$

Dual constraints:

$$
\begin{aligned}
u_1 + u_2 \qquad\qquad + u_5 &\geq 8 \\
u_3 + u_4 \qquad + u_6 &\geq 9 \\
u_1 \qquad + u_3 \qquad\qquad + u_7 &\geq 5 \\
u_2 \qquad + u_4 \qquad\qquad + u_8 &\geq 6 \\
u_i \geq 0, \quad i = 1, \ldots, 8
\end{aligned}
$$

Dual objective:

$$\min \; u_1 + u_2 + u_3 + u_4 - u_9 + u_9 (y_1 + y_2) + u_5 y_1 + u_6 y_1 + u_7 y_2 + u_8 y_2$$

$y_1 = y_2 = 0$ gives dual ray: $d_9 = 1, \; d_i = 0, \; i = 1, \ldots, 8.$

Get constraint: $(d^T H) y \leq b^T d,$

or $\quad -y_1 - y_2 \leq -1 \quad$ or $\quad y_1 + y_2 \geq 1.$

Iteration 2

Master Problem: $\max \; z - 15y_1 - 10y_2$

$\quad$ s.t. $z \leq 28, \; y_1 + y_2 \geq 1$

$\quad y_i$ binary.

Solution: $y = (0, 1), \; z = 28,$ value $= 16.$

Subproblem solution: Multiple solutions with value 11.

$u = (5, 6, 0, 0, 0, 9, 0, 0) \Rightarrow z - 9y_1 \leq 11$

$u = (0, 0, 0, 0, 8, 9, 5, 6) \Rightarrow z - 17y_1 - 11y_2 \leq 0$

$$u=(0,0,5,6,8,0,0,0,0) \Rightarrow z-8y_1 \le 11$$
$$u=(5,0,0,6,3,3,0,0,0) \Rightarrow z-6y_1 \le 11$$
$$u=(0,6,5,0,2,4,0,0,0) \Rightarrow z-6y_1 \le 11.$$

ITERATION 3

Master Problem:

$$\begin{aligned} \max\ & z-15y_1-10y_2 \\ \text{st.}\ & z \le 28 \\ & y_1+y_2 \ge 1 \\ & z-9y_1 \le 11 \\ & z-17y_1-11y_2 \le 0 \\ & y_i \text{ binary} \end{aligned}$$

(Other constraints are weaker)

Solution: $\boxed{y=(1,0)}$, $z=17$, $\boxed{\text{value}=2.}$

Subproblem soln: Multiple solutions with value 17

Eg. $u=(8,0,9,0,0,0,5,6,0)$

Since dual value $=2$, we are optimal.

Find optimal x:

Solve:

$$\begin{aligned} \max\ & 8x_1+9x_2+5x_3+6x_4 \\ \text{st.}\ & x_1+x_3 \le 1 \\ & x_1+x_4 \le 1 \\ & x_2+x_3 \le 1 \\ & x_2+x_4 \le 1 \\ & x_1 \le 1,\ x_2 \le 1,\ x_3 \le 0,\ x_4 \le 0 \\ & x_i \ge 0 \end{aligned}$$

Soln: $\boxed{x_1=x_2=1,}$ value $=17$.

Overall value:

$17-15y_1$ $\boxed{=2}$