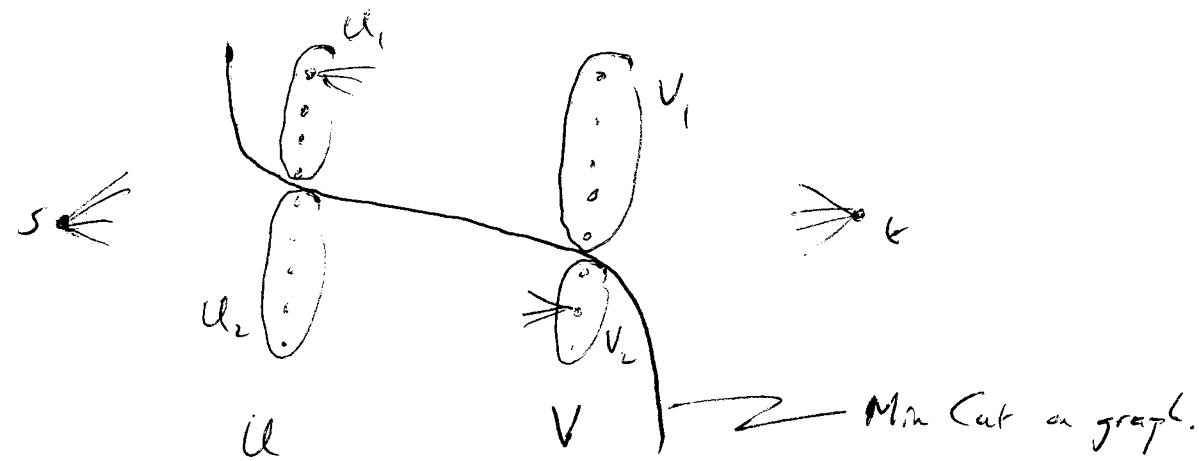
1. Let $G=(U \cup V, E)$ be a bipartite graph, with complement $\overline{G}=(U \cup V, \overline{E})$.

If $W \subseteq U \cup V$ is a set of vertices with the same color in $\overline{G}$ then they must form a clique in G. Hence W is either a single vertex, or a pair of adjacent vertices in G.

Similarly, a clique in $\overline{G}$ is an independent set in G.

Set up a max flow problem on G:
have source s, with arcs of capacity 1 from s to each vertex in U.
have sink t, with arcs of capacity 1 from each vertex in V to t.
each edge in G is an arc directed from U to V with capacity ∞.



Note that $U_2 \cup V_1$ is an independent set in G, so $\omega(\overline{G}) \geq |U_2| + |V_1|$

Also, the MaxFlow gives a matching on G of cardinality $|U_1| + |V_2|$.

Coloring: take edges in matching, plus any unmatched vertices in $U_2 \cup V_1$.
So ~~size of~~ number of colors is $|U_2| \cup |V_1| \geq \chi(\overline{G})$

Hence $\omega(\overline{G}) = \chi(\overline{G})$, since always have $\omega(\overline{G}) \leq \chi(\overline{G})$.

2. Define a matrix Y which is $n \times k$.

Given a partition, let $Y_{ij} = \begin{cases} 1 & \text{if } i \text{ in set } j \\ 0 & \text{otherwise.} \end{cases}$

Let $X = YY^T$, so $X \succeq 0$.

Then X is feasible in the given SDP,

with value equal to objective function value.

Thus, any partition leads to a feasible solution to the SDP,

so the SDP gives a relaxation.

3. Given a tour, define a permutation matrix:

$$X_{ij} = \begin{cases} 1 & \text{if city } i \text{ is in position } j \text{ on tour} \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$(XCX^T)_{kl} = \sum_{i,j} X_{ki} C_{ij} X_{lj}$$

$$= \begin{cases} 2 & \text{if } k,l \text{ adjacent on tour} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\operatorname{tr}(WXCX^T) = 2 \times$ length of tour.

Conversely, a permutation matrix leads to a tour.

4. Need $\begin{vmatrix} x_{11} & x_{13} \\ x_{13} & x_{33} \end{vmatrix} \geq 0$ and $|X| \geq 0$.

$$|X| = x_{11}x_{22}x_{33} + 2x_{12}x_{23}x_{13} - x_{11}x_{23}^2 - x_{22}x_{13}^2 - x_{33}x_{12}^2$$

This is a quadratic function in x_{13} that is concave.

Can differentiate to find the maximum:

$$\frac{d|X|}{dx_{13}} = 2x_{12}x_{23} - 2x_{22}x_{13}$$

Note that if $x_{22}=0$ then $x_{12}=x_{23}=0$ since the two given submatrices are psd. Can then pick $x_{13}=0$ to give a psd X.

If $x_{22} \neq 0$, max is at $x_{13} = \dfrac{x_{12}x_{23}}{x_{22}}$. (Note then $x_{22}>0$.)

Then $$|X| = x_{11}x_{22}x_{33} + 2\frac{x_{12}^2x_{23}^2}{x_{22}} - x_{11}x_{23}^2 - x_{33}x_{12}^2 - \frac{x_{12}^2x_{23}^2}{x_{22}}$$

$$= x_{33}(x_{11}x_{22} - x_{12}^2) + \frac{x_{23}^2}{x_{22}}(x_{12}^2 - x_{11}x_{22})$$

$$= \frac{1}{x_{22}}(x_{22}x_{33} - x_{23}^2)(x_{11}x_{22} - x_{12}^2) \geq 0 \quad ✓$$

Also, $$\begin{vmatrix} x_{11} & x_{13} \\ x_{13} & x_{33} \end{vmatrix} = x_{11}x_{33} - \frac{x_{12}^2x_{23}^2}{x_{22}^2} = \frac{1}{x_{22}^2}\left((x_{11}x_{22})(x_{22}x_{33}) - x_{12}^2x_{23}^2\right)$$

$$\geq 0 \quad ✓$$