1. (a) Assume $x^1, \ldots, x^k$ are linearly independent.
Then the only solution to $\sum_{i=1}^{k} \lambda_i x^i = 0$ is $\lambda_1 = \ldots = \lambda_k = 0$.

So there is no nonzero solution λ to $\sum_{i=1}^{k} \lambda_i x^i = 0$, $\sum_{i=1}^{k} \lambda_i = 0$

So the points are affinely independent.

(b) Prove by contradiction:

Assume $x^1, \ldots, x^k$ are linearly dependent, show they are affinely dependent.

So $\exists\ \lambda_1, \ldots, \lambda_k$ not all zero with $\sum_{i=1}^{k} \lambda_i x^i = 0$.

Thus, $0 = a^T\left(\sum_{i=1}^{k} \lambda_i x^i\right) = \sum_{i=1}^{k} \lambda_i a^T x^i$

$= \sum_{i=1}^{k} \lambda_i b$

$\Longrightarrow \sum_{i=1}^{k} \lambda_i = 0$.

So the points are affinely dependent.

2. ① $x_{ij} + x_{jk} - x_{ik} \leq 1$ is satisfied at equality

if (i) i, j, k all in same cluster
(ii) i, j in one cluster, k in another
(iii) j, k in one cluster, i in another,

with any arrangement of remaining vertices.

Assume $\exists\, g, h$ with $g^T x = h$ for any clustering satisfying ① at equality, and $g^T x \leq h$ a valid inequa

Clustering $\{i,j\}\ \{k\}$, all other vertices on their own
$\Rightarrow \quad g_{ij} = h$ ⓐ

Clustering $\{j,k\}, \{i\}$, all other vertices on their own
$\Rightarrow \quad g_{jk} = h$ ⓑ

Clustering $\{i,j,k\}$, all other vertices on their own
$\Rightarrow g_{ij} + g_{jk} + g_{ik} = h \overset{ⓐⓑ}{\Longrightarrow} g_{ik} = -h.$ ⓒ

Let l, m be any two vertices other than i, j, k.
Clustering $\{ijk\}\ \{lm\}$, everything else on its own
$\Rightarrow g_{ij} + g_{jk} + g_{ik} + g_{lm} = h \Rightarrow \quad g_{lm} = 0$ ⓓ

Let l be any vertex other than i, j, k.
Clustering $\{i,j\}, \{k,l\}$, everything else on its own $\Rightarrow g_{kl} = 0$ ⓔ

Clustering $\{j,k\}, \{i,l\}$, everything else on its own $\Rightarrow g_{il} = 0$ ⓕ

Clustering $\{ijkl\}$, everything else on its own

$$\Rightarrow g_{ij} + g_{ik} + g_{jk} + g_{il} + g_{jl} + g_{kl} = h$$

$$\Rightarrow h + h - h + 0 + g_{jl} + 0 = h$$

$$\Rightarrow g_{jl} = 0 \qquad (g)$$

Finally, valid clustering {all vertices on their own}

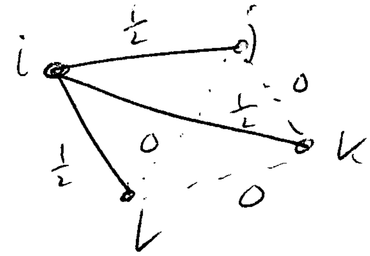
$$\Rightarrow 0 \le h \qquad (h)$$

So the original constraint (or a nonnegative scaling of it)

is the only possibility.

Thus, it is facet defining.

4. Two possibilities:

(I) $x =$ 

Satisfies triangle inequalities.
Cut off by
$$x_{ij} + x_{ik} + x_{il} - x_{jl} - x_{jk} - x_{kl} \leq 1.$$

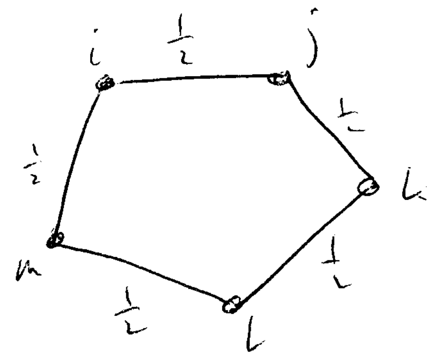
This constraint is valid. Can argue logically about cases.
Alternatively:

Know
$$x_{ij} + x_{ik} - x_{jk} \leq 1$$
$$x_{ij} + x_{il} - x_{jl} \leq 1$$
$$x_{ik} + x_{il} - x_{kl} \leq 1$$

Take $\frac{1}{2}$(each ineq):
$$x_{ij} + x_{ik} + x_{il} - \tfrac{1}{2} x_{jk} - \tfrac{1}{2} x_{jl} - \tfrac{1}{2} x_{kl} \leq \tfrac{3}{2}.$$

Round down:
$$x_{ij} + x_{ik} + x_{il} - x_{jk} - x_{jl} - x_{kl} \leq 1.$$

(II) $x =$



Valid cut:

$$x_{ij} + x_{jk} + x_{kl} + x_{lm} + x_{im} - x_{ik} - x_{il} - x_{jl} - x_{jm} - x_{km} \leq 2. \quad (*)$$

Again, can argue logically by considering cases.
Alternatively:

$$x_{ij} + x_{jk} - x_{ik} \leq 1$$
$$x_{jk} + x_{kl} - x_{jl} \leq 1$$
$$x_{kl} + x_{lm} - x_{km} \leq 1$$
$$x_{lm} + x_{mi} - x_{li} \leq 1$$
$$x_{mi} + x_{ij} - x_{mj} \leq 1$$

Add with multiples of $\frac{1}{2}$:

$$x_{ij} + x_{jk} + x_{kl} + x_{lm} + x_{mi} - \tfrac{1}{2}\left(x_{ik} + x_{jl} + x_{km} + x_{li} + x_{mj}\right) \leq 5/2$$

Rounding down gives (*).