1. (a) $x = (0,0) \Rightarrow t_1 \geq 0,\ t_2 \geq 1 \Rightarrow t_1 + t_2 \geq 1$

$x = (0,1) \Rightarrow t_1 \geq 1,\ t_2 \geq 0 \Rightarrow t_1 + t_2 \geq 1$

$x = (1,0) \Rightarrow t_1 \geq 1,\ t_2 \geq 0 \Rightarrow t_1 + t_2 \geq 1$

$x = (1,1) \Rightarrow t_1 \geq 0,\ t_2 \geq 1 \Rightarrow t_1 + t_2 \geq 1$

(b) $t_1 \geq x_1 - x_2$ (1) $\qquad t_1 \geq -x_1 + x_2$ (2)

$t_2 \geq x_1 + x_2 - 1$ (3) $\qquad t_2 \geq -x_1 - x_2 + 1$ (4)

$$(1)+(3) \Rightarrow 2x_1 - t_1 - t_2 \leq 1$$
$$\Rightarrow x_1 - \tfrac{1}{2}t_1 - \tfrac{1}{2}t_2 \leq \tfrac{1}{2}$$
$$\Rightarrow x_1 - t_1 - t_2 \leq 0 \text{ by C-G rounding}$$
$$\Rightarrow t_1 + t_2 \geq x_1 \quad (5)$$

$$\tfrac{1}{2}\big((2)+(4)+(5)\big) \Rightarrow t_1 + t_2 \geq \tfrac{1}{2}$$
$$\Rightarrow t_1 + t_2 \geq 1 \text{ by C-G rounding.}$$

So the inequality has C-G rank ≤ 2.

To show rank > 1:

Let $P = \{(x_1, x_2, t_1, t_2) : (1),(2),(3),(4) \text{ satisfied, and } x, t \geq 0\}$

The point $(\tfrac{1}{2}, \tfrac{1}{2}, 0, 0) \in P$, so any valid inequality for P must be satisfied by this point.

$(\tfrac{1}{2}, \tfrac{1}{2}, 0, 0) = \tfrac{1}{2}(1,0,0,0) + \tfrac{1}{2}(0,1,0,0)$, so any valid inequality for P must be satisfied by one of these integer points, so any C-G rank 1 ineq must be satisfied by one of them, so can't get $t_1 + t_2 \geq 1$ in one application of C-G rounding.

2. Gomory cutting planes are:

$$\tfrac{1}{8}x_3 + \tfrac{1}{8}x_4 \geq \tfrac{3}{8} \qquad (1)$$

$$\tfrac{3}{4}x_1 + \tfrac{3}{4}x_4 \geq \tfrac{1}{4} \qquad (2)$$

$$(1) \Rightarrow \tfrac{3}{8} \leq \tfrac{1}{8}(4-6x_1+x_2) + \tfrac{1}{8}(7-2x_1-x_2)$$

$$= \tfrac{11}{8} - x_1,$$

or $\boxed{x_1 \leq 1}$

$$(2) \Rightarrow \tfrac{1}{4} \leq \tfrac{3}{4}(4-6x_1+x_2) + \tfrac{3}{4}(7-2x_1-x_2) = \tfrac{33}{4} - 6x_1$$

or $\boxed{x_1 \leq 4/3}$

Strong Gomory cuts:

(3) $\tfrac{1}{8}x_3 + \tfrac{1}{8}x_4 \geq \tfrac{3}{8}$ again, which gives $x_1 \leq 1$.

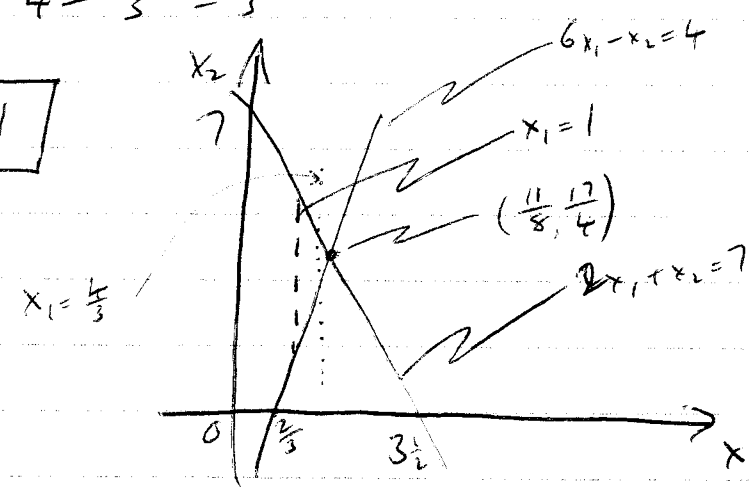
(4) $$x_2 + \left(-1 + \frac{3/4 - 1/4}{1-1/4}\right)x_3 + \left(0 + \frac{3/4 - 1/4}{1-1/4}\right)x_4 \leq 4$$

or $x_2 + \left(-\tfrac{1}{3}\right)x_3 + \tfrac{2}{3}x_4 \leq 4$

or $x_2 - \tfrac{1}{3}(4-6x_1+x_2) + \tfrac{2}{3}(7-2x_1-x_2) \leq 4$

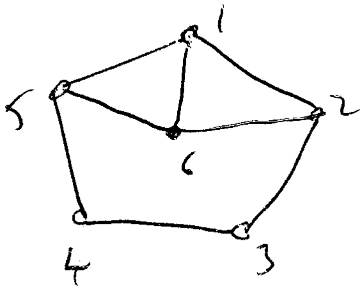
or $\tfrac{2}{3}x_1 \leq 4 - \tfrac{10}{3} = \tfrac{2}{3}$

or $\boxed{x_1 \leq 1}$



4. Two possible configurations:

(i)



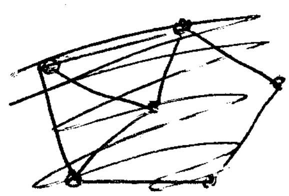
$\max\{\sum_{i=1}^{5} x_i : x_6 = 1,\ x \text{ is node packing}\}$

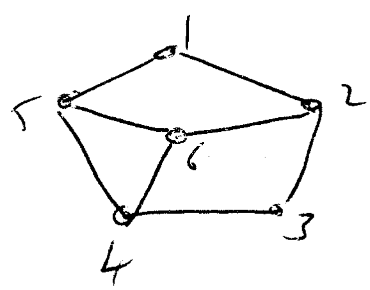
$= 1 = \xi$ (either $x_3 = 1$ or $x_4 = 1$)

so $\alpha = 2 - \xi = 1$

So valid lifted inequality is $\boxed{\sum_{i=1}^{5} x_i + x_6 \leq 2}$

(ii)





$\max\{\sum_{i=1}^{5} x_i : x_6 = 1,\ x \text{ is node packing}\} = 2 = \xi$ $(x_1 = x_3 = 1)$

So $\alpha = 2 - \xi = 0$

So lifted valid inequality is $\boxed{\sum_{i=1}^{5} x_i \leq 2}$.