Thm The MAXCUT polytope is full-dimensional.

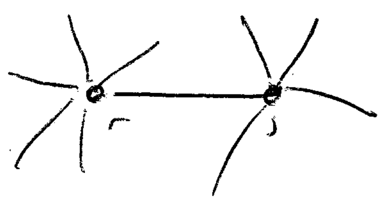
Proof Prove by contradiction.

Assume $\exists$ constraint $\sum_{e \in E} a_e x_e = b$ satisfied by ~~all~~ the incidence vectors of all cuts. Want to show $b = 0$ and $a_e = 0\ \forall$

(1) $b = 0$: $x_e = 0\ \forall e$ is the incidence vector of a cut. So $b = 0$.

(2) $a_e = 0\ \forall e$

Pick edge $\bar{e} = (r, s)$.



Consider three cuts:

(A) r on one side, $V \setminus r$ on other:

$$a_{rs} + \sum_{\substack{e \in \delta(r) \\ e \neq \bar{e}}} a_e = 0 \qquad (i)$$

(B) s on one side, $V \setminus s$ on other:

$$a_{rs} + \sum_{\substack{e \in \delta(s) \\ e \neq \bar{e}}} a_e = 0 \qquad (ii)$$

(C) r, s on one side, $V \setminus \{r, s\}$ on the other:

$$\sum_{\substack{e \in \delta(r) \\ e \neq \bar{e}}} a_e + \sum_{\substack{e \in \delta(s) \\ e \neq \bar{e}}} a_e = 0 \qquad (iii)$$

$(i) + (ii) - (iii) \Rightarrow 2a_{rs} = 0 \Rightarrow a_{\bar{e}} = 0$. True $\forall e \in E$. $\blacksquare$

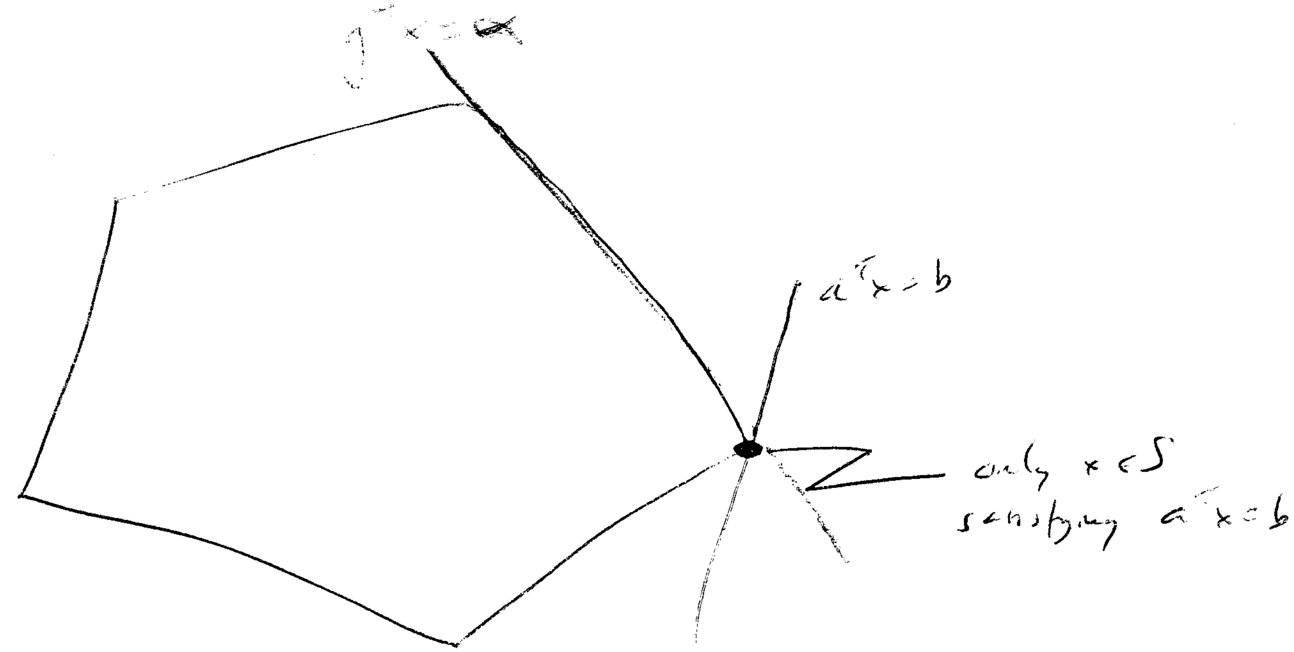
Note:

This proof technique can be extended to prove that an inequality is facet defining.

For simplicity, assume S is full-dimensional.

Let $a^T x \le b$ be a valid inequality.

Facet defining $\Leftrightarrow$ $\nexists$ another ~~equ~~ valid inequality $g^T x \le \alpha$ with (g, α) not a multiple of (a, b), and $g^T x = \alpha$ for any $x \in S$ satisfying $a^T x = b$



Eg: $x_{ij} - x_{ik} - x_{jk} \le 0$ (*) is a facet defining inequality:

This inequality is satisfied at equality by cuts $\{i,j,k\}, \emptyset$; $\{i,k\}, \{j\}$; $\{k,j\}, \{i\}$ with the other vertices assigned randomly.

Assume our inequality is implied by $g^T x \le h$.

Consider the empty cut: $V, \emptyset$. Thus $h = 0$

Consider the cut $V \setminus l, l$ ($l \neq k$): $\sum_{v \in V} g_{vl} = 0$ (a)

Consider the cut $V \setminus \{l,m\}, \{l,m\}$, with $l, m \notin \{i,j,k\}$:

$$\sum_v g_{vl} + \sum_v g_{vm} - 2 g_{lm} = 0 \overset{(a)}{\Longrightarrow} g_{lm} = 0. \quad (b)$$

Consider the cut $V \setminus \{j,l\}, \{j,l\}$, with $l \notin \{i,k\}$:

$$\sum_v g_{vj} + \sum_v g_{vl} - 2 g_{lj} = 0 \overset{(a)}{\Longrightarrow} g_{lj} = 0 \quad (c)$$

Similarly, $V \setminus \{i,l\}, \{i,l\}$ with $l \notin \{j,k\}$ shows $g_{li} = 0$. (d)

Still need to find g_{ij}, g_{ik}, g_{jk}, and g_{lk} for $l \notin \{i,j\}$.

Consider again $V \setminus i, i$: $0 = \sum_{v \in V} g_{vi} = \sum_{v \in V, v \notin \{i,j,k\}} g_{vi} + g_{ij} + g_{ik}$

$$\overset{(d)}{\Longrightarrow} g_{ij} + g_{ik} = 0 \quad (e)$$

Similarly, $V \setminus j, j \Longrightarrow g_{ij} + g_{jk} = 0$ (f)

$\Rightarrow g_{ik} = g_{jk} = -g_{ij}$ (g)
So important coefficients agree with (*).

Finally, look at g_{lk} for $l \notin \{i,j\}$:

$V \setminus ik \Rightarrow \sum_v g_{iv} + \sum_v g_{kv} - 2 g_{ik} = 0 \Rightarrow g_{ji} + g_{kj} + \sum_{v \in V, v \neq i,k,j} g_{vk} = 0 \Rightarrow \sum_{v \in V, v \neq i,j,k} g_{vk} = 0$ (h)

$V \setminus ikl$, $l \notin \{i,j,k\}$ $\Rightarrow 0 = \sum_v g_{iv} + \sum_v g_{kv} + \sum_v g_{lv} - 2 g_{ik} - 2 g_{il} - 2 g_{kl} \overset{(d),(h),(b)}{=} g_{ij} + g_{jk} - 2 g_{kl} \overset{(g)}{=} -2 g_{kl}$

MAXCUT problem.

$G=(V,E)$. Edge costs c_e.

Divide V into two sets V_1, V_2 with $V_1 \cup V_2 = V$,
$V_1 \cap V_2 = \phi$.

Want to maximize $\sum_{\substack{e=i,j \\ i \in V_1 \\ j \in V_2}} c_e$.

Model with $x_e = \begin{cases} 1 & \text{if edge is in cut.} \\ 0 & \text{otherwise} \end{cases}$

Eg:

Thm Every cycle and every cut intersect in an even number of edges.

Eg:

Get ineqs $x(F) - x(C \setminus F) \leq |F| - 1$ (*)

for every cycle C, ~~cut set F~~ subset F of C, $|F|$ odd.

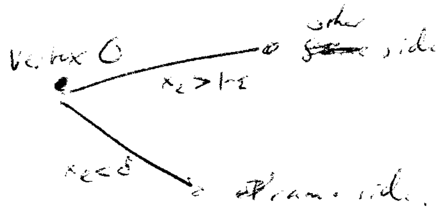
Facet defining provided C is a chordless cycle.

Eg:

How do we find violated constraints?

Heuristic (Barahona, Jünger, Reinelt, also ... for Ising spin glasses):

Breadth first search:



Grow tree until find vertex we've already seen.

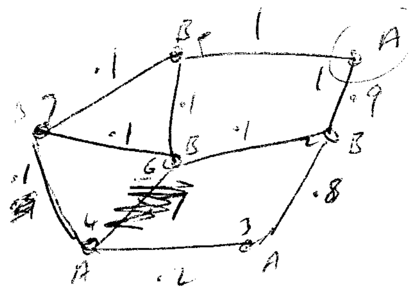
Hopefully, this gives a violated inequality.

Exact: (Barahona-Mahjoub):

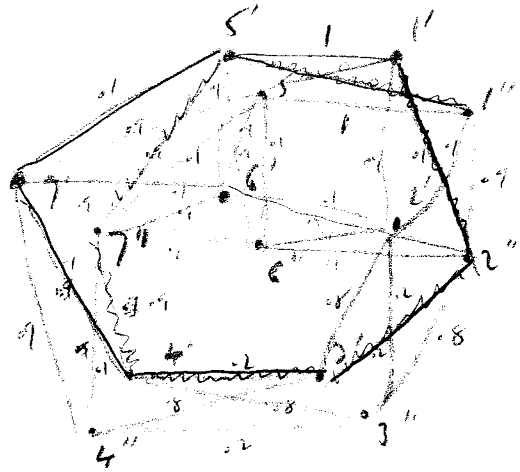
Form a new graph G' with two vertices i', i'' for each old vertex i.
for each $(i,j) \in G$: edges $i'j'$, $i''j''$ have weight $x(ij)$
edges $i'j''$, $i''j'$ have weight $1-x(ij)$.

Find shortest path from i' to i''.
If path has length < 1, then cycle ineq. is violated

Eg:



~~Grow~~ Search for here:



Length $= .1 + .2 + .2 + .1 + .1 + 0$
$= .7 < 1.$
(This may not be the shortest path)

Integer programming formulation:

$$\max \sum c_e x_e$$

$$\text{s.t. } x(F) - x(C \setminus F) \le |F| - 1$$

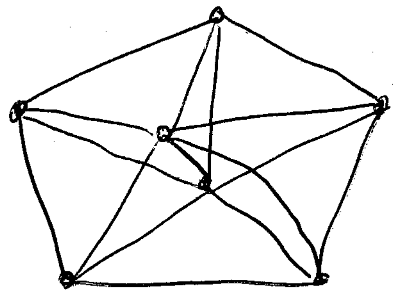
for every cycle C, subset $F \subseteq C$, $|F|$ odd

x binary.

Solution to LP relaxation may not be integer.

Other constraints are needed in the description of conv(S).

Eg: Bicycle wheel:



$n+2$ vertices, n odd.

Can have at most $2n$ of these edges in a cut – central vertices on one side, rim vertices on the other.

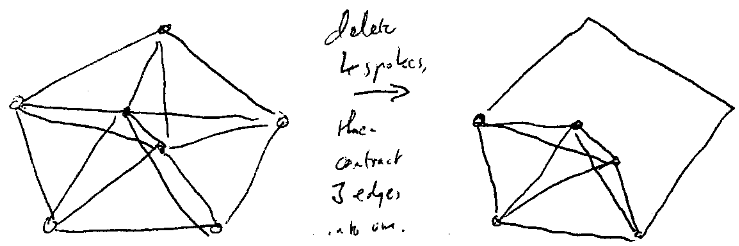
If all $x_{ij} = \frac{2}{3}$ then all odd set ineqs are satisfied.

We have $3n+1$ edges, so $x_{ij} = \frac{2}{3}$ has weight $2n + \frac{2}{3} > 2n$.

So the constraint $\sum_{e \in \text{Wheel}} x_e \le 2n$ is valid, and not implied by the odd set constraints.

Note: If G contains no K_5 minor (eg, G is planar), then the ~~odd~~ cycle inequalities (*) suffice.

Bicycle wheel has a K_5-minor:



Linear Ordering

Grötschel, Jünger, Reinelt, OR 1984

Given n objects, with $\bar{c}_{ij}$ = cost of having i before j,

find the best ordering of the objects.

NP-Complete.

Let $\bar{x}_{ij} = \begin{cases} 1 & \text{if } i \text{ before } j \\ 0 & \text{otherwise.} \end{cases}$

Can express as:

$$\min \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \bar{c}_{ij} \bar{x}_{ij}$$

s.t.

$\cancel{\bar{x}_{ij} + \bar{x}_{ji} = 1}$

$\cancel{\bar{x}_{ij} \geq 0}$

$\bar{x}$ incidence vector of ordering.

Note that $\bar{x}_{ij} + \bar{x}_{ji} = 1$. So eliminate half the variables: Let $x_{ij} = \bar{x}_{ij}$, $c_{ij} = \bar{c}_{ij} - \bar{c}_{ji}$, $1 \leq i < j \leq n$,

$$\min \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{ij} x_{ij}$$

s.t. x incidence vector of ordering.

Can express as an IP:

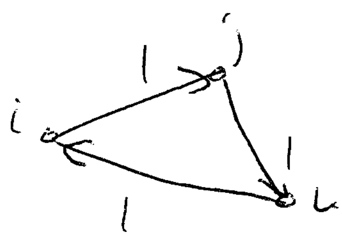
$$\min \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} c_{ij} x_{ij}$$

$$\text{s.t.} \quad x_{ij} + x_{ji} = 1 \quad \text{for all } i,j$$

$$x_{ij} + x_{jk} + x_{ki} \leq 2 \quad \text{for all } i,j,k$$

$$x_{ij} = 0 \text{ or } 1.$$

The inequalities $x_{ij} + x_{jk} + x_{ki} \leq 2$ rule out



— impossible for an ordering.

Can show:

(i) Dim (lin ordering polytope) $= \dfrac{n(n-1)}{2}$ (so only necessary equalities are $x_{ij} + x_{ji} = 1$) These are lin indep since each variable only appears in one of them

(ii) $x_{ij} \geq 0$ and $\underbrace{x_{ij} \leq 1}$ are facets of lin ordering polytope
This is only a facet if we drop half the variables.

(iii) $x_{ij} + x_{jk} + x_{ki} \leq 2$ is a facet of lin ordering polytope.

(iv) $x_{ij} + x_{jk} + x_{kl} + x_{li} \leq 3$ is not a facet.

(v) There are other classes of facets.

~~Proofs of (ii):~~ (iii):

Proofs of (i), (ii) rest on showing that get right number of affinely independent points.

Do this by only looking at x_{ij} for $1 \le i < j \le n$.

~~Can get orderings with~~

Consider the orderings:

6	5	4	3	2	1	$x = 0$
6	5	4	3	1	2	$x = [1\ 0\ 0 \ldots 0]$
6	5	4	1	3	2	
6	5	1	4	3	2	
6	1	5	4	3	2	
1	6	5	4	3	2	
1	6	5	4	2	3	
1	6	5	2	4	3	
1	6	2	5	4	3	
1	2	6	5	4	3	
1	2	6	5	3	4	
1	2	6	3	5	4	
1	2	3	6	5	4	
1	2	3	6	4	5	
1	2	3	4	6	5	
1	2	3	4	5	6	

gives $5+4+3+2+1+1 = 15+1$

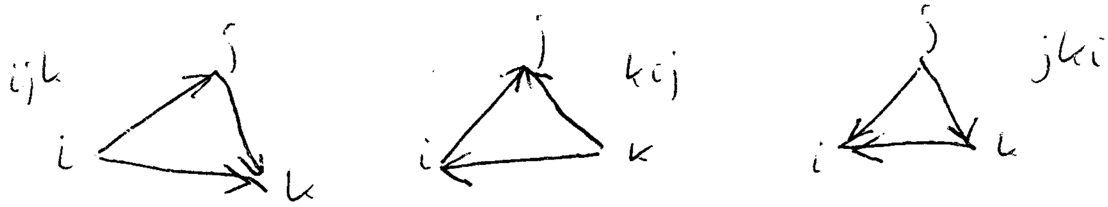
$= \frac{6 \times 5}{2} + 1$

aff indep orderings in polytope ✓.

Can't do the last flip if have $x_{65} = 1$.

So face has $\frac{n(n-1)}{2}$ affinely indep

(ii)
Proof of $x_{ij} + x_{jk} + x_{ki} \le 2$ ~~is~~ a facet:



	ij	jk	ki
ijk	1	1	0
kij	1	0	1
jki	0	1	1

lin indep.

NOTE:

	ijk	kij	jki
ji	0	0	1
ki	0	1	1
kj	0	1	0

$x_{..} + x_{12} - x_{13} \le 2.$

	1234	1243	1423
→ 12	1	1	1
← 13	1	1	1
14	1	1	1
→ 23	1	1	0
24	1	1	0
34	1	0	0

Sufficient to look at half the variables:

	1234	1243	1423	4123	4312	4231
21	0	0	0	0	0	1
31	0	0	0	0	1	1
32	0	0	0	0	1	0
41	0	0	0	1	1	1
42	0	0	1	1	1	1
43	0	1	1	1	1	1

With this choice of variables,
easy to see affine independence.

	12345	12354	12534	15234	51234	51243	51423	54123	54312	54231
21	0	0	0	0	0	0	0	0		
31	0	0	0	0	0	0	0	0		
32	0	0	0	0	0	0	0	0		
41	0	0	0	0	0	0	0	1		
42	0	0	0	0	0	0	1	1		
43	0	0	0	0	0	1	1	1		
51	0	0	0	0	1	1	1	1		
52	0	0	0	1	1	1	1	1		
53	0	0	1	1	1	1	1	1		
54	0	1	1	1	1	1	1	1		

4 orderings from adjusting position of 5

3 from adjusting pos of 4.

3 orderings for different perms of 1, 2, 3.

In general get:

$$\underbrace{(n-1)}_{\text{for } n} + \underbrace{(n-2)}_{\text{for } n-1} + \dots + \underbrace{3}_{\text{for } 4} + \underbrace{1}_{\text{last one}} + \underbrace{2}_{\text{for orderings of } 1,2,3.} = \frac{n(n-1)}{2}$$

✓ so have fact.

(iv)

$$x_{12} + x_{23} + x_{31} \leq 2$$

$$x_{13} + x_{34} + x_{41} \leq 2$$

$$x_{12} + x_{23} + 1 + x_{34} + x_{41} \leq 4$$

$$\Rightarrow x_{12} + x_{23} + x_{34} + x_{41} \leq 3 \quad \checkmark.$$