KP 1

Knapsack Problem

max $c^T x$

s.t. $a^T x \le b$ Let N = set of indices

x binary.

~~Let C be a subset of N satisfy~~

Assume $b > 0$, $a > 0$, $c > 0$, $a_j \le b$

So $\mathrm{conv}(S)$ is full dimensional.

Let $C \subseteq N$ satisfy:

$$\sum_{j \in C} a_j > b \qquad (C \text{ is a cover})$$

$$\sum_{\substack{j \in C \\ j \ne k}} a_j \le b \quad \forall k \in C \qquad (\text{and so } C \text{ is a minimal cover})$$

Eg: $3x_1 + 4x_2 + 5x_3 + x_4 + 2x_5 \le 11$.

Then $C = \{\cancel{4}, 1, 2, 3\}$ works.

Then $\sum_{j \in C} x_j \le |C| - 1$ is valid. Facet defining for $X' = \{x \in \{0,1\}^{|C|} : \sum_{j \in C} a_j x_j \le b\}$ here.

Need to lift it to get a facet:

~~Let~~

Solve $\max \sum_{j \in C} x_j : \cancel{x_j} \sum_{j \in C} a_j x_j \le b - a_k$ for $k \notin C$

Then add k to C, repeat until $\cancel{S}\, N \setminus C = \emptyset$.

See over for the example

$$3x_1 + 4x_2 + 5x_3 + x_4 + 2x_5 \leq 11.$$

Have constraint $\quad x_1 + x_2 + x_3 \leq 2.$

Want to lift it.

Lift first on x_4:

$$\begin{aligned} \max \quad & x_1 + x_2 + x_3 \\ \text{s.t.} \quad & 3x_1 + 4x_2 + 5x_3 \leq 11 - 1 = 10 \\ & x_i \text{ binary}, \; i = 1, 2, 3 \end{aligned}$$

Optimal value is 2 still.

So x_4 comes in with coefficient 0

Now lift on x_5:

$$\begin{aligned} \max \quad & x_1 + x_2 + x_3 \\ \text{s.t.} \quad & 3x_1 + 4x_2 + 5x_3 + x_4 \leq 11 - 2 = 9 \\ & x_i \text{ binary}, \; i = 1, 2, 3, 4. \end{aligned}$$

Optimal value is 2 still.

So x_5 also comes in with coefficient 0.

So lifted constraint is $\quad x_1 + x_2 + x_3 \leq 2.$

This is facet defining, since it was facet defining for $x_1 - x_2 - x_3$ space, and the dimension of the polyhedron increased by one at each lifting, and we had a maximal lifting.

See bottom of next page for another example.

KP2.

Also: Can extend earlier idea:

$C \subseteq N$, $\sum_{j \in C} a_j \le b$ (~~What is this~~?)

$k \in N \setminus C$,

if $Q \cup \{k\}$ is a minimal cover $\forall Q \subseteq C$ for $|Q| = k$ ~~$2 \le |Q| \le |C|$~~ for some k satisfying $2 \le k \le |C|$.

then have valid inequality

$$(r-k+1)x_k + \sum_{j \in T(r)} x_j \le r$$

So if $x_k = 1$ then at most $k-1$ of the elements in $T(r)$ can be in the knapsack.

where $T(r) \subseteq C$, $|T(r)| = r$, $k \le r \le |C|$

Example on back:

Eg (of lifting a minimal cover)

Constraint

$$8x_1 + 3x_2 + 3x_3 + 4x_4 \le 9$$

$\{2,3,4\}$ is minimal cover $x_2 + x_3 + x_4 \le 2$ valid.

Lift. max $x_2 + x_3 + x_4$: $3x_2 + 3x_3 + 4x_4 \le 9 - 8 = 1$, x_i binary

So max $= 0$.

Lifted ineq: $2x_1 + x_2 + x_3 + x_4 \le 2$.

Note: have to solve a knapsack problem to find the lifting coefficient!

Matching: (not perfect matching)

$\sum x_e \le 1$

Full dimensional (origin + unit vectors)

Odd set ineq:

w/ edges 1, 2, 3:

for each edge not in a triangle, can combine with one edge in triangle

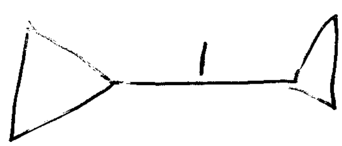
$$\begin{bmatrix}1\\0\\0\\0\\\vdots\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\\0\\\vdots\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\\0\\\vdots\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\\1\\0\\\vdots\\0\end{bmatrix}$$

1, 2 indep.

$\Rightarrow$ all indep.

Trickier with perfect matching, to even get the dimension.

Eg



7 edges, so $\subseteq \mathbb{R}^7$

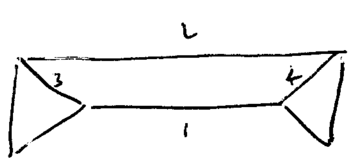
6 degree constraints.

But $\dim(\text{feas region}) = 0 < 7 - 6$.

So have implicit equalities: eg: $x_1 \le 1$

becomes $x_1 = 1$.

Eg



Now for two feasible solns, 8 edges, 6 degree constraints.

Implicit equalities include $x_3 = 0$, $x_4 = 0$, $x_1 + x_2 = 1$.

Resolution cuts for Satisfiability

$$y_1 \vee y_2 \vee \bar{y}_3 \quad , \quad y_2 \vee y_3 \vee y_4$$

For both of these to be true, need

y_1 or y_2 or y_4 to be true.

This is RESOLUTION.

If express SAT problem as an IP:

Get constraints

$$y_1 + y_2 + (1-y_3) \geq 1 \quad (1)$$
$$y_2 + y_3 + y_4 \geq 1 \quad (2)$$

Add: $y_1 + 2y_2 + y_4 + 1 \geq 2$

$\Leftrightarrow \quad -y_1 - 2y_2 - y_4 \leq -1$

$\Leftrightarrow \quad -\frac{1}{2}y_1 - y_2 - \frac{1}{2}y_4 \leq -1/2$

$\Rightarrow \quad -y_1 - y_2 - y_4 \leq -1$ (round down)

$\Rightarrow \quad y_1 + y_2 + y_4 \geq 1 \quad (3)$

When does this cut off the feasible point to (1), (2)?

Eg: $\bar{y}_2 = .5$, $\bar{y}_3 = .5$, $\bar{y}_1 = \bar{y}_4 = 0$.

(— to indicate we are considering a particular point.)

In general: ~~$\bar{y}_2$~~

If $[\bar{y}_1 + \bar{y}_2 + (1-\bar{y}_3)]$

$+ [\bar{y}_2 + \bar{y}_3 + \bar{y}_4] \geq 3$

then $\bar{y}_1 + 2\bar{y}_2 + \bar{y}_4 \geq 2$

so $2\bar{y}_1 + 2\bar{y}_2 + 2\bar{y}_4 \geq 2$

so $\bar{y}_1 + \bar{y}_2 + \bar{y}_4 \geq 1$ already.

So definitely need $\boxed{\text{sum of lhs}_{\text{of (1),(2)}} < 3.}$

If sum of $\text{lhs}_{\text{of (1),(2)}} = 2 + \alpha$ and $\bar{y}_2 \leq \alpha$ then

$[\bar{y}_1 + 2\bar{y}_2 + \bar{y}_4 \geq 1 + \alpha \Rightarrow \bar{y}_1 + \bar{y}_2 + \bar{y}_4 \geq 1]$

So need $\boxed{\text{common term large relative to sum of lhs}}$

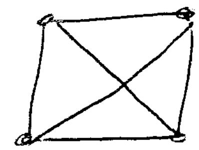
~~Cycle ineqs~~

Clique inequalities: (Steve Joy)

For 2 SAT:

$x_i \vee x_j \quad \forall\, i,j \in V \subseteq \{1,\dots,n\}.$

Need $\sum_{i\in V} x_i \geqslant |V| - 1.$



Vertices correspond to elements o[f]

Edges correspon[d] to clauses.

Need an ~~vertex~~ edge cover made up of vertices.

MAX 2SAT:

$\min \sum w_i y_i$

s.t. $\cancel{x_i + x_j}$ $l_i + l_j + y_k \geqslant 1$ for each clause $k = \{l_i \vee l_j\}$

~~$x_i + (1-x_j) + y_k \geqslant 1$~~ ~~for each clause $k = \{x_i \vee \bar{x}_j\}$~~

~~$(1-x_i) + (1-x_j) + y_p \geqslant 1$~~ ~~for each clause $p = \{\bar{x}_i \vee \bar{x}_j\}$.~~

l_i binary, y_k binary, where l_i is a literal.

~~x_i, y_k binary.~~

l_i related to $\bar{l}_i$

CREATE DESCRIP[TION] IS BET[TER]

We may have already combined these ineqs to and have:

eg:

~~$l_i + l_j + y$~~

$$\left.\begin{aligned}
x_1 + x_2 &+ y_1 &\geqslant 1\\
x_1 + x_3 &+ y_1 + y_2 &\geqslant 1\\
x_1 + x_4 &+ y_3 &\geqslant 1\\
x_2 + x_3 &+ y_1 + y_4 &\geqslant 1\\
x_2 + x_4 &+ y_2 + y_5 &\geqslant 1\\
x_3 + x_4 &+ y_6 &\geqslant 1
\end{aligned}\right\}$$

Think of deleting edges when set $y_i = 1$.

Get ineq:

$x_1 + x_2 + x_3 + x_4 + \cancel{p_1 y_1 + p_2}$ $\sum_{i=1}^{6} p_i y_i \geqslant 3$

p_i are lifting coeffs

Take $p_1 = 3$, $p_2 = 2$, $p_3 = p_4 = p_5 = p_6$

May be able to do better.

$x_1+x_2+y_1 \geq 1$

$x_1+x_3+y_1+y_2 \geq 1$

$x_1+x_4+y_3 \geq 1$

$x_2+x_3+y_1+y_4 \geq 1$

$x_2+x_4+y_2+y_5 \geq 1$

$x_3+x_4+y_6 \geq 1.$

If $y_i=0\ \forall i$: must have $x_1+x_2+x_3+x_4 \geq 3$ (Not valid for other choices of y)

If $y_1=1$, all other $y_i=0$: Still need $x_1+x_4 \geq 1$

$x_2+x_4 \geq 1$

$x_3+x_4 \geq 1$

So need $x_1+x_2+x_3+x_4 \geq 1.$

So lift to $x_1+x_2+x_3+x_4+2y_1 \geq 3$

If $y_2=1$: ~~Might allow $y_1=0$~~, and then any

$\min(x_1+x_2+x_3+x_4+2y_1 : y_2=1, y_3=\cdots=y_6=0) = 1$

so lift to $x_1+x_2+x_3+x_4+2y_1+2y_2 \geq 3$

If $y_3=1$: min is 2, so $p_3=1$.

Also get $p_4=p_5=p_6=1$