Strong valid inequalities for structured integer programs

Motivate by considering node packing problem:

Given a graph $G=(V,E)$, a node packing is a ~~co~~ subset $U \subseteq V$ such that ~~$v_i \in U, v_j \in U \Rightarrow (v_i, v_j) \notin E$.~~

no pair in the set is joined by an edge.

So, setting $x_i = \begin{cases} 1 & \text{if node } i \text{ is in packing} \\ 0 & \text{o/w} \end{cases}$

~~so~~ ~~to~~ we have

$$S = \{x \in B^n : x_i + x_j \leq 1 \text{ for all } (i,j) \in E\}. \quad (n = |V|)$$

S contains $n+1$ affinely independent vectors:

The origin.

The n unit vectors.

$\therefore \dim(\text{conv}(S)) = n$

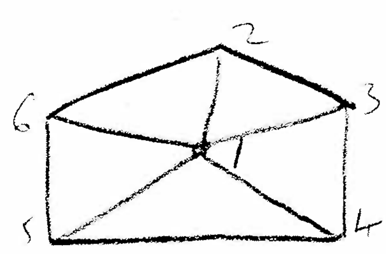
$C \subseteq V$ is a <u>clique</u> if each pair of nodes in C is joined by an edge.

~~Ex~~ Any node packing can include at most one vertex from any clique.

So get valid ineqs for ~~S~~ S:

Clique constraint $\rightarrow$ $\sum_{j \in C} x_j \leq 1$ for any clique C of G.

Eg



A maximal clique is a clique C such that $C \cup \{i\}$ is not a clique for any $i \in V \setminus C$.

So for the example, maximal clique constraints are:

$$\begin{array}{llllll} x_1 + x_2 + x_3 & & & & \leq 1 \\ x_1 & + x_3 + x_4 & & & \leq 1 \\ x_1 & & + x_4 + x_5 & & \leq 1 \\ x_1 & & & + x_5 + x_6 & \leq 1 \\ x_1 + x_2 & & & + x_6 & \leq 1 \end{array} \qquad (*)$$

Maximal clique constraints define facets of conv(S):

A facet is a face of dimension $n-1$ $\quad$ (dim (conv(S)) $= n$).

So we give n affinely indep pts in S which satisfy $\sum_{j \in C} x_j = 1$:

For $j \in C$:
Consider the packing consisting purely of vertex j.

For $j \notin C$:
Since C is maximal, $\exists\, l(j) \in C$ such that $(j, l(j)) \notin E$.

Consider the packing consisting of j and $l(j)$.

The incidence vectors of these ~~pa~~ n packings are aff indep, so the clique constraint $\sum_{j \in C} x_j = 1$ defines a facet.

In example:
Consider $C = \{1, 2, 3\}$.

Then get node packings $\quad \{1\} \quad \{2\} \quad \{3\} \quad \{4,2\}, \{5,2\}, \{6,3\}$

Incidence vectors:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Cols ~~Rows~~ are lin indep, so certainly aff indep. So points are aff indep.

Show $x_i \geq 0$ is facet defining: requires showing aff indep — can't get lin indep vectors 0 + n unit vectors

Now consider point $\bar{x} = \frac{1}{2}(0\ 1\ 1\ 1\ 1\ 1)^T$.

This satisfies all ~~co~~ clique constraints, but it is not in conv(S) –

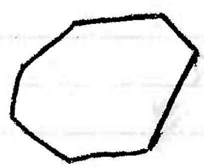
it is an extreme point of polyhedron defined by $(*)$ and nonnegativity.

To cut off $\bar{x}$, consider <u>odd hole</u> inequalities:

Suppose $\exists\ H \subseteq V$, $|H|$ odd, vertices of H can be ordered $(i_1, i_2, \ldots, i_p)$

such that $(i_r, i_s) \in E$ iff $s = r+1$ or $s = 1$ and $r = p$.

Then H is an <u>odd hole</u>



If H is an odd hole, then

$$\sum_{j \in H} x_j \le \frac{|H|-1}{2}$$ is satisfied by all node packings.

So in example, $H = \{2,3,4,5,6\}$ is an odd hole, and we obtain the constraint

$$x_2 + x_3 + x_4 + x_5 + x_6 \le 2 \qquad (1)$$

This cuts off $\bar{x}$.

Does (1) define a facet? No – it gives a face of dimension 4.

The packings $\{2,4\}, \{2,5\}, \{3,5\}, \{3,6\}, \{4,6\}$ satisfy (1) at equality,

and they give aff indep. vectors. But $\nexists$ sixth packing which satisfies (1) at equality.

We have a face but not a facet.

Maybe (1) can be improved by tilting it to get a facet.

Consider the ineq

$$\alpha x_1 + (x_2 + x_3 + x_4 + x_5 + x_6) \leq 2. \qquad (2)$$

For what values of α is this ineq valid for S?

Need to consider what happens when $x_1 = 0$ and when $x_1 = 1$.

If $x_1 = 0$, (2) is valid for ~~S~~ S for any α.

If $x_1 = 1$, any packing has $x_2 = x_3 = x_4 = x_5 = x_6 = 0$

So (2) is valid for S for $\alpha \leq 2$.

$\therefore$ get ineq

$$2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 2 \qquad (3)$$

valid for S.

(3) is a facet for S: the five packings given above, together with ~~ϵ~~ $\{1\}$ ~~define~~ ~~define~~ have characteristic vectors which satisfy (3) at equality, and the char vectors are affinely independent.

This is an illustration of a general procedure ~~called~~ called <u>lifting</u>.

$$\begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\\1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\\1\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\\0\\1\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\\0\\0\\1\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\\0\\1\end{bmatrix}$$

Incidence vectors of packings satisfying

$2x_1+x_2+x_3+x_4+x_5+x_6 = 2.$

To show they are aff indep., sufficient to show they are lin indep.

LI:

$\lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4 \quad \lambda_5 \quad \lambda_6$

$x_1: 0 \Rightarrow \lambda_1 = 0$

$$\left.\begin{aligned}\lambda_2+\lambda_3&=0\\ \lambda_2+\lambda_6&=0\\ \lambda_3+\lambda_4&=0\\ \lambda_4+\lambda_5&=0\end{aligned}\right\} \Rightarrow \lambda_2=\lambda_4=-\lambda_3=-\lambda_6=-\lambda_5$$

$\lambda_5+\lambda_6=0 \Rightarrow \lambda_i = 0$

Break into two propositions and a corollary

<u>Proposition</u> Suppose $S \subseteq B^n$

Let $S^0 = S \cap \{x \in B^n : x_1 = 0\}$

$S^1 = S \cap \{x \in B^n : x_1 = 1\}$

Assume $\sum_{j=2}^{n} \pi_j x_j \leq \pi_0$ (4) is valid for S^0.

Let $\zeta = \max \{ \sum_{j=2}^{n} \pi_j x_j : x \in S^1 \}$.

Then

$$\alpha x_1 + \sum_{j=2}^{n} \pi_j x_j \leq \pi_0 \qquad (5).$$

is valid for S for any $\alpha \leq \pi_0 - \zeta$.

Overestimating ζ (e.g. by solving LP relaxation) still gives a valid inequality

another proposition: Moreover if $\alpha = \pi_0 - \zeta$ and if $\sum_{j=2}^{n} \pi_j x_j \leq \pi_0$ defines a face of S^0 of dimension k

then $\alpha x_1 + \sum_{j=2}^{n} \pi_j x_j \leq \pi_0$ defines a face of S of dimension at least $k+1$;

Make this a corollary? ~~If~~ & and if $\sum_{j=2}^{n} \pi_j x_j \leq \pi_0$ defines a facet of S^0 then

$\alpha x_1 + \sum_{j=2}^{n} \pi_j x_j \leq \pi_0$ defines a facet of S, provided $\dim(S) = \dim(S^0) + 1$.

~~(No proof)~~

NB: Proposition gives another way for showing

~~Lifting is done sequentially.~~

a valid ineq defines a facet: show that it can be obtained as a maximum lifting of a facet of a lower dimensional ~~facet~~ polyhedron.

<u>Proof</u> To show (5) is valid:

If $x \in S^0$:

$$\alpha x_1 + \hat{\sum_{j=2}} \pi_j x_j = \hat{\sum_{j=2}} \pi_j x_j \leq \pi_0 \quad \text{from (4)}$$

If $x \in S^1$:

$$\alpha x_1 + \hat{\sum_{j=2}} \pi_j x_j = \alpha + \hat{\sum_{j=2}} \pi_j x_j \leq \alpha + \varsigma$$
$$\leq \pi_0 - \varsigma + \varsigma = \pi_0$$

To show (5) is a face of dimension at least $k+1$ if (4) is a face of S^0 of dimension k:

Since (4) gives a face of dimension k of S^0, $\exists$ $k+1$ aff indep pts in S^0 which satisfy

$$\hat{\sum_{j=2}} \pi_j x_j = \pi_0.$$

Let these points be $x^1, x^2, \dots, x^{k+1}$.

Note that $x_1^i = 0$, since $x^i \in S^0$.

Now consider the point x^* which solves

$$\max \{ \pi^T x : x \in S^1 \}$$

Then $\alpha x_1^* + \sum_{j=2}^{n} \pi_j x_j^* = \pi_0$, and $x_1^* = 1$.

Since $x_1^* = 1$, x^* cannot be expressed as an affine combination of $x^1, \dots, x^{k+1}$.

Hence the points $x^*, x^1, \dots, x^{k+1}$ are affinely independent. //

Proof of facet part:

(4) defines a facet of S^0.

Hence (5) defines a face of dimension $\geq \dim(S^0) = \dim(S) - 1$

of S. Since (4) is a facet of S^0, ~~but not~~ $\exists\, x^0 \in S^0$

such that $\sum_{j\geq 2}^{\hat{n}} \pi_j x_j < \pi_0$.

Then $x^0 \in S$ is such that

$$\alpha x_1^0 + \sum_{j\geq 2}^{\hat{n}} \pi_j x_j < \pi_0,$$

so (5) does not contain S, so (5) defines a facet of S. //

Lifting procedure should be used sequentially:

Have ineq

$$\sum_{j\in N'} \pi_j x_j \leq \pi_0$$

valid for $S \cap \{x \in B^n : x_j = 0, j \in N \setminus N'\}$

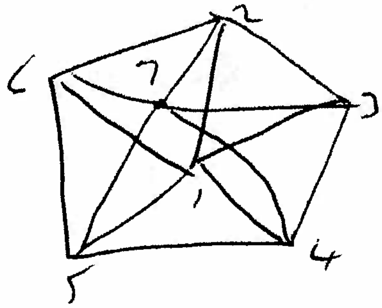
Lift variables in $N^* \setminus N'$ one at a time to get

$$\sum_{j\in N\setminus N'} \alpha_j x_j + \sum_{j\in N'} \pi_j x_j \leq \pi_0$$

valid for S.

See over for eg's:

Eg 1: Lifting order matters:

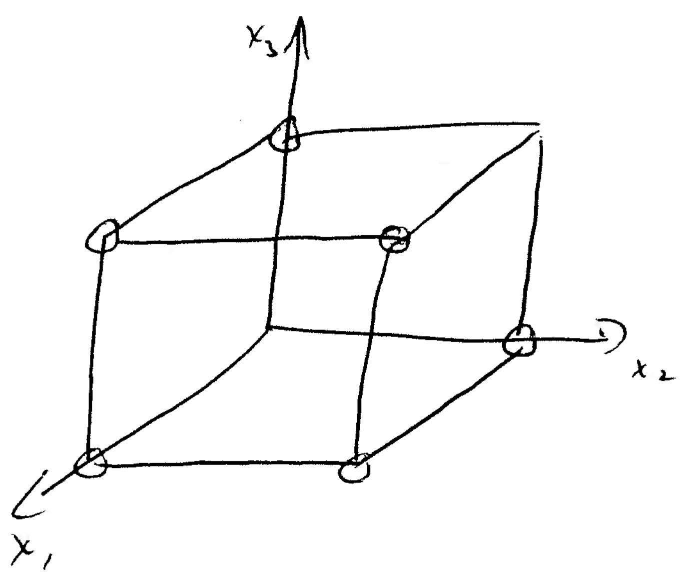


$x_2 + \cdots + x_6 \le 2$

Lift on x_1 then x_7 gives $2x_1 + x_2 + \cdots + x_6 \le 2$

Lift on x_7 then x_1 gives $x_2 + \cdots + x_6 + 2x_7 \le 2$.

Eg 2: Dimension of face can increase by more than 1:



$x_2 \le 1$ has dimension 0 in S^0

Lift to $x_2 \le 1$, with dimension 2 in S.