Also: B&B in parallel
w/ coarse grain parallel machine

Options in Branch-and-Bound

__Branching strategy__: How do we decide which node to examine next?

Having a good feasible solution is important – can prune more nodes by bounds.

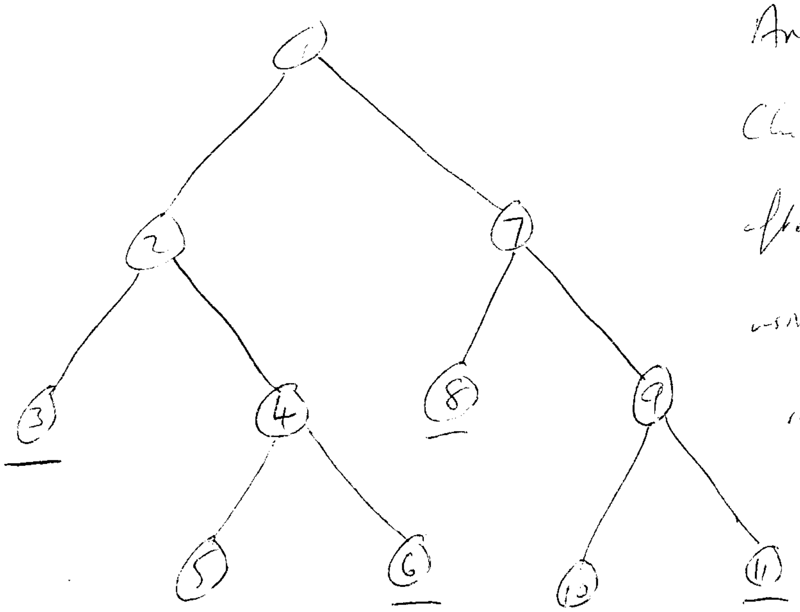
From experience, it seems that the best way to get a feasible solution is to use depth-first-search:

If the current node is not pruned, ~~one of its children~~ ~~is~~ the next node considered is one of its children.

After pruning, use __backtracking__ to get next node:

Go back on path from current node to root until find the first node which has a child which has not been pruned. The next node considered is that child.

Eg



Another advantage of dfs: Child is considered immed after its parent, so ~~easy to~~ if using simplex, can easily reoptimize w/o having to reinvert basis.

Alternatives to depth-first-search with backtracking:

Could use bounds:

$\bar{z}^i$ = upper bound on $z(P^i)$ obtained from parent of P^i

Could choose P^i with largest $\bar{z}^i$.

Strong branching: [illegible]

Relaxation strategy

LP relaxation — drop integrality requirements

Other possibilities:

i) If x binary, omit $Ax \le b$.

Relaxation trivially solved (Balas)

Try to fathom by checking if infeasible, examining each

constraint $a_i^T x \le b_i$ in turn

ii) Lagrangean relaxation (Geoffrion, Fisher, Held-Karp)

Each (P_j^i) is of form

$$\begin{aligned} \max \quad & c^T x \\ & Ax \le b \quad \leftarrow \text{nice constraints (eg. network constraints)} \\ & Dx \le e \quad \leftarrow \text{nasty constraints} \\ & x \in T \quad \leftarrow \text{includes integrality \& bounds} \end{aligned}$$

Omitting $Dx \le e$ gives easy problem

Consider "Lagrangean relaxation" for given $\lambda \ge 0$, consider

[illegible] p. 58

$$\max_{(P_i(\lambda))} \quad c^T x + \lambda^T (e - Dx) = \varphi(\lambda).$$
$$Ax \le b$$
$$x \in T$$

(Choose λ to minimize φ)

Separation strategies

Take $S = S^1 \cup S^2$ with

$S^1 = S \cap \{x \in \mathbb{R}^n_+ : dx \le d_0\}$,

$S^2 = S \cap \{x \in \mathbb{R}^n_+ : dx \ge d_0 + 1\}$.

where d and d_0 are integer.

If x^* is soln to $\max\{c^T x : x \in \mathbb{R}^n_+, Ax \le b\}$,

choose (d, d_0) so that $d_0 < d^T x^* < d_0 + 1$.

Then x^* is not feasible in either of the subsequent relaxations.

Most common choice of (d, d_0) is:

Variable dichotomy:

Choose nonintegr $x_j^* = t$, separate

$x_j \le \lfloor t \rfloor$ $\qquad$ $x_j \ge \lfloor t \rfloor + 1$

Which variable do we choose?
See ~~Padberg~~, Crowder, Johnson, Padberg, OR 31, 1983, pp 803-834.

An alternative: or ZOOM: Singhal, Marsten, Morin, ORSA J. of C. 1(1), 1989, pp
– fixed order branching.

If we have generalized upper bound (GUB) constraint

$$\sum_{j \in Q} x_j = 1$$

for some $Q \subseteq N$ (multiple choice constraint); then can split by

$$\sum_{j \in Q_1} x_j = 0 \qquad \sum_{j \in Q \setminus Q_1} x_j = 0$$

$Q_1 \subseteq Q$ $\qquad$ $Q_1 \subseteq Q$

x^* is infeasible in relaxations provided $0 < \sum_{j \in Q_1} x_j^* < 1$.

Implicit Enumeration. (N&R, p. 175)

Schemes for computing bounds.

Use simple feasibility and cost comparisons, obtained by direct inspection of a problem's constraints and objective function.

Eg: $\max \quad x_1 + 2x_2 - 3x_3 - x_4$

$$\text{s.t.} \quad -3x_1 - 3x_2 - 3x_3 + x_4 \le -5$$
$$2x_1 + 4x_2 + 5x_3 + 4x_4 \le 5.$$
$$x_i \text{ binary}, \; i = 1, \dots, 4.$$

First constraints implies we need at least two of the variables equal to 1.
Second constraint implies we can have at most one of the variable equal to 1.
Thus, this ~~sub~~problem is infeasible.
Note: LP relaxation is feasible: $x = (\frac{5}{6}, \frac{5}{6}, 0, 0)$.

Can use other procedures. ~~Ed~~ Also useful when deciding which node to examine next.

~~Strong branching.~~

Pseudocosts: (Parker & Rardin, p. 200) (The IP is a MINIMIZATION problem)

Estimate value ~~cost~~ of child

Have ~~relaxed~~ problem $P(S_k)$, relaxation $P(\bar{S}_k)$ solved to optimality, so know $v(P(\bar{S}_k))$.

Should we branch on 0 or 1?

So need to forecast $v(P(S_k : x_p = 0))$ and $v(P(S_k : x_p =$

Assign penalties for variables that are currently fractional.

Eg:

$$v(P(S_k : x_p = 0)) \approx v(P(\bar{S}_k)) + \mu \bar{x}_p^k + \mu \sum_{\substack{j \in I \\ j \neq p}} \min\{\bar{x}_j^k, 1 - \bar{x}_j^k\}$$

$$v(P(S_k : x_p = 1)) \approx v(P(\bar{S}_k)) + \mu (1 - \bar{x}_p^k) + \mu \sum_{\substack{j \in I \\ j \neq p}} \min\{\bar{x}_j^k, 1 - \bar{x}_j^k\}.$$

More sophisticated:

$$v(P(S_k : x_p = 0)) \approx v(P(\bar{S}_k)) + \mu_p^- \bar{x}_p^k + \sum_{\substack{j \in I \\ j \neq p}} \min\{\mu_j^- \bar{x}_j^k, \mu_j^+ (1 - \bar{x}_j^k)\}$$

separate μ for each variable and also for increasing or decreasing the variable.

See overhead.

When setting up S_8 and S_9:

How do we get μ?

Use earlier information. Ie, what happened before when we branched?

Still simple μ:

$$\mu = \left(\cancel{4} \sum_{\text{all previous branches}} \frac{\text{Increase in value of relaxation}}{\text{Change in fixed variable}} \right) \Big/ \# \text{ previous branches.}$$

$$= \frac{1}{6} \left(\sum_{\text{six numbers}} \frac{4.2-3.6}{8/9} + \frac{5.0-3.6}{1/9} + \frac{5.3-4.2}{8/9} + \frac{5.1-4.2}{1/9} + \frac{5.2-5.0}{1/9} + \frac{5.8-5.0}{1/9} \right) \approx 4.22$$

Estimate of pseudo-cost for nodes 8, 9:

$$8: \; v(P(S_8)) \approx 5.1 + 4.22\left(\tfrac{2}{5}\right) = 6.79 \;\leftarrow$$

$$9: \; v(P(S_9)) \approx 5.1 + 4.22\left(\tfrac{3}{5}\right) = 7.63.$$

Can also estimate:

$\mu_2^+ =$ per unit cost of changing x_2 from fractional to 1

$$\approx \frac{5.8-5.0}{8/9} \approx 0.90$$

$\mu_2^- =$ per unit cost of changing x_2 from fractional to 0

$$\approx \frac{5.2-5.0}{1/9} \approx 1.80$$

Pseudo cost for nodes 8, 9:

$$v(P(S_8)) \approx 5.1 + \tfrac{2}{5}(1.80) = 5.82$$

$$v(P(S_9)) \approx 5.1 + \tfrac{3}{5}(0.90) = 5.64 \;\leftarrow$$

Parker & Rardin, p. 200 →

(MINIMIZATION problem).

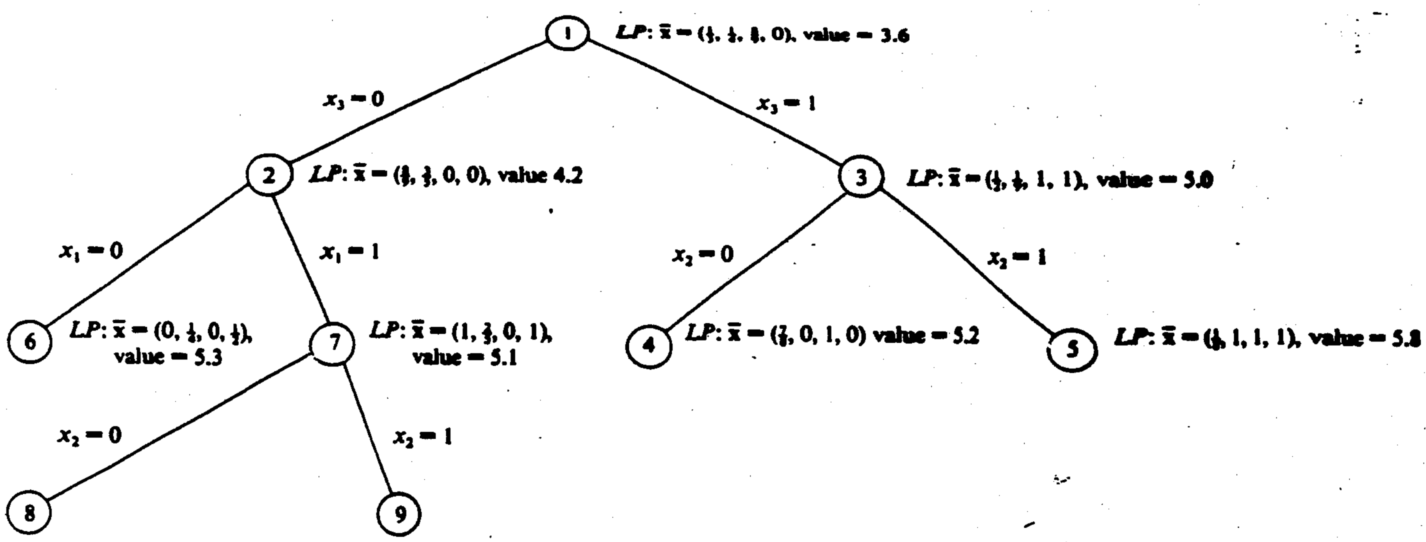


Fig. 5.6. Pseudo-cost Example 5.9.

Variable fixing:

$$\min\ c^T x$$
$$\text{s.t. } Ax = b$$
$$x \geq 0, \text{ integer.}$$

Say $x_i = 0$ in optimal solution to current relaxation.
Let $\bar{c}_i$ be its reduced cost.
Let z_{LB} be the value of the relaxation.
Let z^{UB} be the value of a known feasible integer solution.

If $x_i = 1$ then the value is at least $z_{LB} + \bar{c}_i$.

So: $\boxed{\text{if } z_{LB} + \bar{c}_i > z^{UB} \text{ then can fix } x_i = 0 \text{ in this branch of the tree}}$

Can extend to fixing $x_i = 1$ if appropriate.

Can extend to interior point methods:

Current relaxation

$$\min\ c^T x$$
$$\text{s.t. } Ax = b$$
$$x \geq 0$$

Dual

$$\max\ b^T y$$
$$\text{s.t. } A^T y + s = c$$
$$s \geq 0$$

Have feasible solution $\bar{x}, \bar{y}, \bar{s}$ to relaxation.
Now, $c^T x = b^T y + x^T s$ for any x.
So if $b^T \bar{y} + \bar{s}_i > z^{UB}$, can fix $x_i = 0$.