Branch-and-Bound

B&B is an example of an enumerative algorithm. Uses idea of divide and conquer.

Idea: Want to solve the integer program

$$\max\ z_{IP} = c^T x \qquad \qquad (P).$$
$$Ax \leq b$$
$$x \text{ binary}$$

Let $S = \{x \in \mathbb{B}^n : Ax \leq b\}$.

Divide S into

$$S^0 = \{x \in S : x_1 = 0\}$$
$$S^1 = \{x \in S : x_1 = 1\}.$$

Now $S = S^0 \cup S^1$, $S^0 \cap S^1 = \phi$.

Look at the two IPs:

$$\max\ c^T x =: z^0 \quad (P^0) \qquad \max\ c^T x =: z^1 \quad (P^1)$$
$$x \in S^0 \qquad \qquad x \in S^1.$$

Then

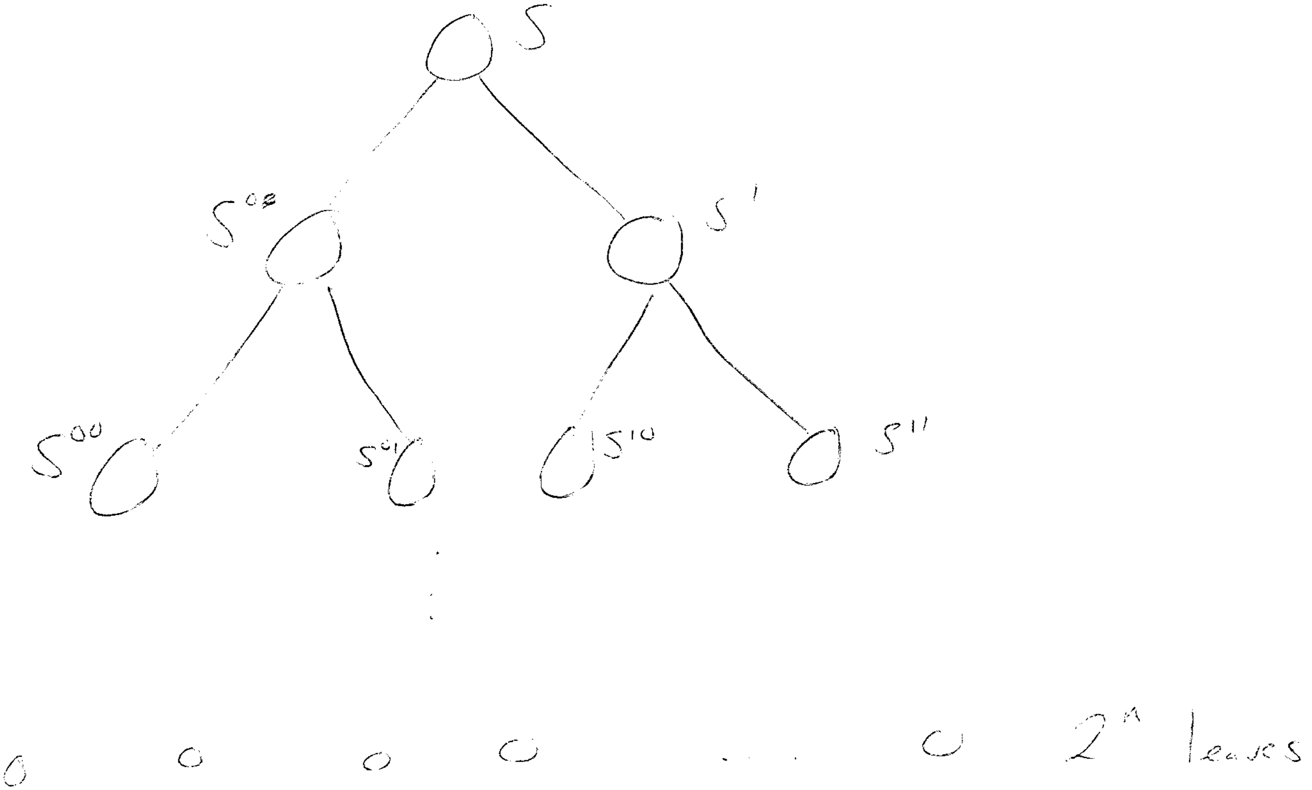
$$z_{IP} = \max\{z^0, z^1\}$$

How do we solve (P^0), (P^1)?

Consider further subdivision:

$$S^{00} = \{x \in S : x_1 = 0,\ x_2 = 0\}$$
$$S^{01} = \{x \in S : x_1 = 0,\ x_2 = 1\} \qquad \text{etc.}$$

Get enumeration tree:



Compare all feasible leaves ~ best gives optimal soln.

How can we limit search?

~~Let $z^i = \max\{c^T x : x \in S^i\}$~~

Let P^i be the problem $\max_{x \in S^i} z^i = c^T x$

i) if P^i is infeasible then all of its descendants will be infeasible, so can stop branching at node S^i.

ii) If P^i is easily solved: solve it, don't need to solve descendants separately.

iii) If P^i has optimal value $z^i < \bar{z} = c^T\bar{x}$ for some known $\bar{x}$ feasible for ~~S~~ P, don't branch further: no point, can't do any better than $\bar{x}$ by pursuing this branch further.

Conditional clauses are hard to verify, so consider $(\bar{P}^i)$, a relaxation of P^i (same obj fn, more feas solns).

Let $\bar{z}^i$ be optimal solution of $\bar{P}^i$. Then alternative conditions for pruning are:

i) $\bar{P}^i$ infeasible

ii) An optimal soln to $\bar{P}^i$ is a feasible soln to P^i (so it must be optimal for P^i.)

iii) Bounds: $\bar{P}^i$ has optimal value $\bar{z}^i < \bar{z} = c^T\bar{x}$ for some known $\bar{x}$ feasible for P.

One possible relaxation of P^i is the LP-relaxation: omit integrality condition.

We say ~~P~~ ~~is~~ we prune at node S^i because of

$$\left\{\begin{array}{l}\text{infeasibility}\\ \text{feasibility}\\ \text{bounds}\end{array}\right. \quad \text{in} \quad \left\{\begin{array}{l}\text{i)}\\ \text{ii)}\\ \text{iii)}\end{array}\right. .$$

Branch and Bound Example

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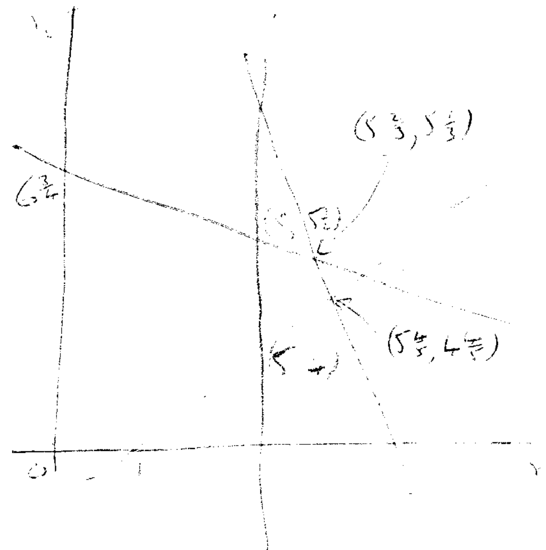
Example:

$$\max \quad 2x_1 + 5x_2$$
$$4x_1 + x_2 \leq 28$$
$$x_1 + 4x_2 \leq 27$$
$$x_1 - x_2 \leq 1$$
$$x_1, x_2 \geq 0 \text{, integral}$$



Soln to LP relaxation:

$x_1 = 5\frac{2}{3}$, $x_2 = 5\frac{1}{3}$, value 38

Split on x_1

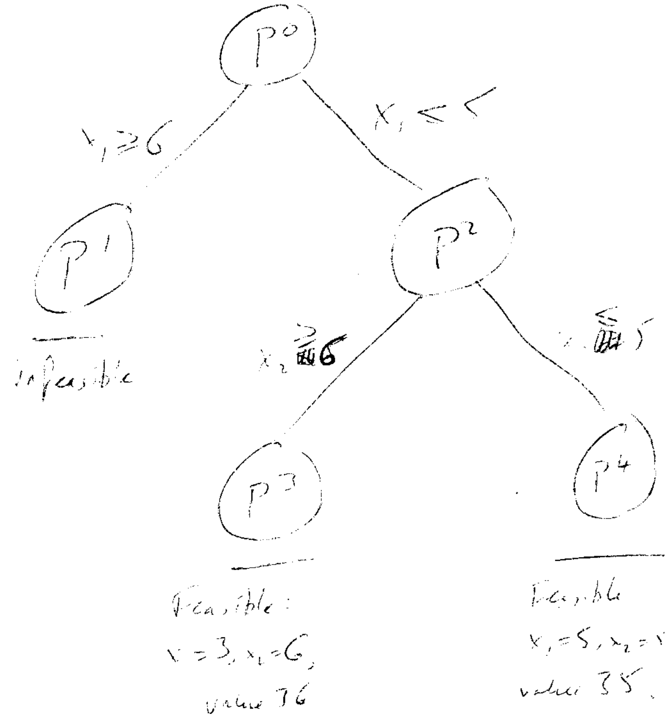
$x_1 \geq 6$: $\quad \max \ 2x_1 + 5x_2$

$$(P^1) \qquad \begin{aligned} 4x_1 + x_2 &\leq 28 \\ x_1 + 4x_2 &\leq 27 \\ x_1 - x_2 &\leq 1 \\ x_1 &\geq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Infeasible

$x_1 \leq 5$ $\quad \max \ 2x_1 + 5x_2$

$$(P^2) \qquad \begin{aligned} 4x_1 + x_2 &\leq 28 \\ x_1 + 4x_2 &\leq 27 \\ x_1 - x_2 &\leq 1 \\ x_1 &\leq 5 \end{aligned}$$



Optimal $x_1 = 5$, $x_2 = 5\frac{1}{2}$, value $37\frac{1}{2}$

Split on x_2

$x_2 \geq 6$: max $2x_1 + 5x_2$

(P^3)

$$
\begin{aligned}
4x_1 + x_2 &\leq 28 \\
x_1 + 4x_2 &\leq 27 \\
x_1 - x_2 &\leq 1 \\
x_1 &\leq 5 \\
x_2 &\geq 6 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Optimal soln : $x_1 = 3$, $x_2 = 6$, value 36

Pruned by feasibility.

$x_2 \leq 5$: max $2x_1 + 5x_2$

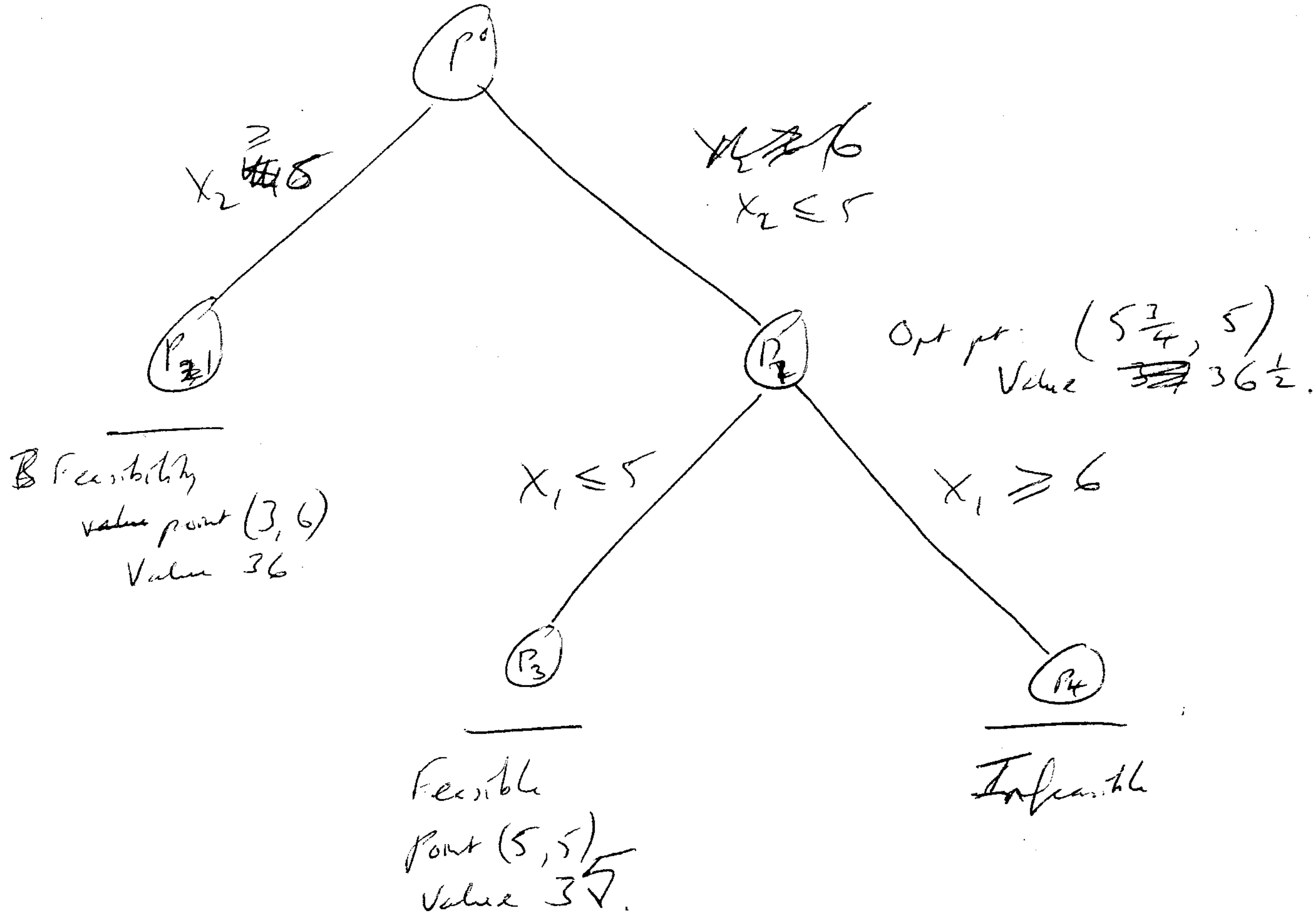
$$
\begin{aligned}
4x_1 + x_2 &\leq 28 \\
x_1 + 4x_2 &\leq 27 \\
x_1 - x_2 &\leq 1 \\
x_1 &\leq 5 \\
x_2 &\leq 5 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Optimal soln : $x_1 = 5$, $x_2 = 5$, value 35

Pruned by bound ~~$~~ (or feasibility)

So optimal soln is $\underline{\underline{x_1 = 3,\ x_2 = 6}}$.

Branch on x_2 first:



Get algorithm for general IPs:

$$\max \quad z_{IP} = c^T x$$

$$\begin{aligned} Ax &= b \\ 0 \leq x &\leq u \\ x &\text{ integer} \end{aligned} \qquad (P) \quad \cancel{(P^0)}$$

Algorithm:

(0) If have feas soln $\bar{x}$ (incumbent) for (P), set $\bar{z} \leftarrow c^T \bar{x}$ (lower bound on optimal value)

Else $\bar{z} \leftarrow -\infty$

(1) Branch Choose some unfathomed, unselected problem (active node) $(P^i_?)$
[If none; $\bar{z} > -\infty \Rightarrow \bar{x}$ optimal, $\bar{z} = -\infty \Rightarrow (P)$ infeas.]

(2) Relax Let $\bar{P}^i$ be relaxation of (P^i). Let opt value be
$\bar{z}^i \; (\geq \underbrace{z(P^i)}_{\text{opt value of } P^i})$

(3) Prune? If $\bar{z}^i = -\infty$ (i.e. $(\bar{P}^i)$ infeas)

Prune by infeasibility

If $\bar{z}^i \leq \bar{z}$, prune by bounds

If $\bar{z}^i \geq \bar{z}$ and opt soln x_i $(\bar{P}^i)$, feas in (P^i),

$\bar{x} \leftarrow x_i$, $\bar{z} \leftarrow \bar{z}^i$, prune by feasibility

If able to prune, go to (1).

(4) Separation If P ~~Choose some unexpanded node~~ of enumeration ~~tree, where~~
If unable to prune: relaxation has been ~~solved~~. Split feasible region for this vertex into two disjoint