Benders Decomposition:

NγW II.3.7.
(NγW III.5.4)
Wolsey, Chap. 11, exercise 5.

Reduce a problem with n continuous variables and p integer variables to subproblems with l continuous var and p integer vars.

$$\max_{x,y} \; c^T x + d^T y$$

$$\text{s.t.} \; Ax + Hy \le b \qquad (P)$$

$$x \ge 0 \qquad y \in Y$$

(NγW uses $\max \; c^T x + h^T y$ s.t. $Ax + Gy \le b$, $x \in \mathbb{Z}^n_+$, $y \in \mathbb{R}^p_+$)

For each fixed $\bar{y}$:

$$\max_x \; c^T x \; (+ d^T \bar{y})$$

$$\text{s.t.} \; Ax \le b - H\bar{y} \qquad (P(\bar{y})) \quad \text{LP in } x.$$

$$x \ge 0.$$

Dual: $$\min_u \; (b - H\bar{y})^T u \; (+ d^T \bar{y})$$

$$\text{s.t.} \; A^T u \ge c \qquad (D(\bar{y}))$$

$$u \ge 0.$$

Note that feasible region of $(D(\bar{y}))$ is same for all choices of y.

Thus, (P) is equivalent to

$$\max_{y \in Y} \Big\{ d^T \bar{y} + \min_{\substack{u : A^T u \ge c \\ u \ge 0}} (b - H\bar{y})^T u \Big\}$$ Gives (MP) immediately.

Let $\bar{u}$ be a ~~optimal~~ feasible solution to $(D(\bar{y}))$.

Then we know, ~~that~~ by weak duality,

$$c^T x (+ d^T \bar{y}) \leq (b - H\bar{y})^T \bar{u} + d^T \bar{y}$$

ie $$c^T x \leq (b - H\bar{y})^T \bar{u}$$

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True for all $\bar{y}$, since ~~all~~ $\bar{u}$ is feasible in all $(D(\bar{y}))$

So suggests problem

$$\begin{aligned} \max \quad & z + d^T y \\ \text{s.t.} \quad & z \leq (b - Hy)^T \bar{u} \qquad \forall \bar{u} \text{ satisfying } A^T\bar{u} \geq c,\ \bar{u} \geq 0. \\ & y \in Y. \end{aligned}$$

ie $$\begin{aligned} \max \quad & z + d^T y \\ \text{s.t.} \quad & z + (\bar{u}^T H) y \leq b^T \bar{u} \qquad \forall \bar{u} \text{ satisfying } A^T\bar{u} \geq c,\ \bar{u} \geq 0. \\ & y \in Y \end{aligned}$$

Really only need to consider extreme points $u^1, \ldots, u^p$ and extreme rays $r^1, \ldots, r^q$ of $\{u : A^T u \geq c, u \geq 0\}$:

$$\begin{aligned} \max \quad & z + d^T y \\ \text{s.t.} \quad & z + (u^{k^T} H) y \leq b^T u^k \quad k = 1, \ldots, p \qquad (MP) \\ & (r^{l^T} H) y \leq b^T r^l \quad l = 1, \ldots, q \\ & y \in Y. \end{aligned}$$

Why this constraint for rays?

If r^j is a ray:

For $P(\bar{y})$ to have feasible solution, need $D(\bar{y})$ to have finite optimal ~~solution~~ value.

Ie, need $(b - H\bar{y})^T r^j \geq 0$ for $j = 1, \ldots, q$.

ie, $(r^{j^T} H)^T \bar{y} \leq b^T r^j$.

Why does cut generated cut off current y?

BENDERS DECOMPOSITION ALGORITHM

$$\begin{array}{lll} \max & c^T x + d^T y & \\ \text{s.t.} & Ax + Hy \le b & (P) \\ & x \ge 0 \quad y \in Y & \end{array}$$

Let K = index set of extreme points of $\{u \ge 0 : A^T u \ge c\}$
J = index set of extreme rays of $\{u \ge 0 : A^T u \ge c\}$.

(1) INITIALIZE with some subset $\bar{K} \subseteq K$, $\bar{J} \subseteq J$.

(2) Solve

$$(MP) \quad \begin{array}{lll} \max\limits_{z,y} & z + d^T y & \\ \text{s.t.} & z + (u^{k^T} H) y \le b^T u^k & \forall k \in \bar{K} \\ & r^{j^T} H y \le b^T r^j & \forall j \in \bar{J} \\ & z \text{ scalar}, \ y \in Y. & \end{array}$$

Let $\bar{z}$, $\bar{y}$ be solution.

(3) Solve

$$\begin{array}{lll} \min & (b - H\bar{y})^T u & \\ \text{s.t.} & A^T u \ge c & (SP) \\ & u \ge 0 & \end{array}$$

If solution is unbounded: add extreme ray to $\bar{J}$, return to (2)

If solution has finite value $< \bar{z}$: add extreme point to $\bar{K}$, return to (2)

If solution has finite value $= \bar{z}$: STOP, optimal.

Algorithm

Let K = index set of extreme points of $\{u \geq 0 : A^T u \geq c\}$

J = index set of extreme rays of $\{u \geq 0 : A^T u \geq c\}$.

1. Initialize with some subset $\bar{K} \subseteq K$, $\bar{J} \subseteq J$.

2. Solve

$$\max_{z,y} \; z + d^T y$$
$$\text{s.t.} \quad z + (u^{k^T} H) y \leq b^T \cancel{g} u^k \quad k \in \bar{K}$$
$$(r^{j^T} H) y \leq b^T r^j \quad j \in \bar{J}.$$
$$y \in Y$$

Let $(\bar{z}, \bar{y})$ be solution.

$(\bar{z}, \bar{y})$ is optimal $\Leftrightarrow$
$\bar{z} + (u^{k^T} H)\bar{y} \leq b^T u^k \; \forall k \in K$, $(r^{j^T} H)\bar{y} \leq b^T r^j \; \forall j \in J$
ie $\Leftrightarrow \bar{z} \leq (b - H\bar{y})^T u^k \; \forall k \in K$, $(b - H\bar{y})^T r^j \geq 0 \; \forall j \in J$

3. Solve

$$~~\max~~ \min \; (b - H\bar{y})^T u$$
$$\text{s.t.} \quad A^T u \geq c$$
$$u \geq 0.$$

(Note: z is our estimate for this of the contribution to the objective from x. If this overestimates the true contribution, need to constrain y and z further.)

If solution is unbounded, add extreme ray to $\bar{J}$, return to 2.

If solution has ~~value~~ finite value $< \bar{z}$, add extreme point to $\bar{K}$, return to 2.

If solution has finite value $= \bar{z}$, STOP: optimal.

Solve $P(\bar{y})$ to find optimal x.

EXAMPLE OF BENDERS' DECOMPOSITION

max $\quad x_1 + 4x_2 + 10y_1$

s.t. $\quad x_1 + x_2 + y_1 \le 2 \qquad (P)$

$\quad -x_1 + 2x_2 + 5y_1 \le 3$

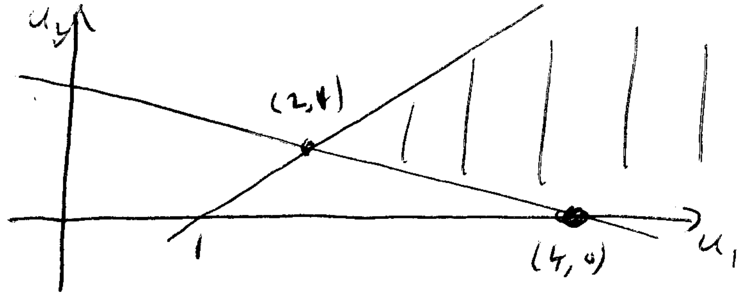
$\quad x \ge 0, \quad y$ binary

Dual feasible region:

$u_1 - u_2 \ge 1$

$u_1 + 2u_2 \ge 4$

$u_i \ge 0$



Initialize with just $u^1 = (4,0)$. Then $u^{1T}H = 4$, $u^{1T}b = 8$.
$u^2 = (2,1)$. Then $u^{2T}H = 7$, $u^{2T}b = 7$.

Revised Master Problem: max $z + 10y_1$

s.t. $z + 4y_1 \le 8$

$z + 7y_1 \le 7$

y_1 binary.

Optimal soln: $y_1 = 1$, $z = 0$, value $= 10$.

Dual problem with $y_1 = 1$:

min $\quad u_1 - 2u_2$

s.t. $\quad u_1 - u_2 \ge 1$

$u_1 + 2u_2 \ge 4$

$u_i \ge 0$

Solution unbounded, with ray $r^1 = (1,1)$.
Get $r^{1T}H = 6$, $r^{1T}b = 5$

Revised Master Problem: max $z + 10y_1$

s.t. $z + 4y_1 \le 8$

$z + 7y_1 \le 7$

$6y_1 \le 5$

Soln: $z = 7, y_1 = 0$

Dual problem has objective $2u_1 + 3u_2$.
Minimized at $u_1 = 2, u_2 = 1$, value 7.

$\rightarrow y_1 = 0$
Solve (P) for $y_1^* = 0$ to find:
$x_1^* = \frac{1}{3}$,
$x_2^* = \frac{5}{3}$.

Benders 5.