Name:

MATP6640/DSES6770
Linear Programming
Spring 2008

Midterm Exam, Thursday, April 3, 2008.

Please do all three problems. Show all work. No books or calculators allowed. You may use any result from class, the homeworks, or the texts, except where stated. You may use one sheet of handwritten notes. The exam lasts 110 minutes.

Q1	/40
Q2	/30
Q3	/30
Total	/100

Throughout, the standard primal-dual LP pair is

$$\begin{array}{llll} \min & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array} \quad (P) \qquad\qquad \begin{array}{llll} \max & b^T y & & \\ \text{s.t.} & A^T y + s & = & c \\ & s & \geq & 0 \end{array} \quad (D)$$

where A is $m \times n$ and the vectors are dimensioned appropriately.

1. (40 points)

   (a) (20 points) **Use complementary slackness** to show that $y = (10, 5)$ solves the linear program

$$\begin{array}{lrcrcl} \max & 11y_1 & + & 8y_2 & & \\ \text{s.t.} & 5y_1 & & & \leq & 50 \\ & & & 5y_2 & \leq & 50 \\ & 3y_1 & + & 4y_2 & \leq & 50 \\ & 2.5y_1 & + & 4.33y_2 & \leq & 50 \end{array} \qquad (D_4)$$

   (Hint: $2.5 \times 10 + 4.33 \times 5 = 46.65$.)

   (b) (20 points) The problem in part (1a) is a relaxation of the true problem,

$$\begin{array}{llcll} \max_y & 11y_1 & + & 8y_2 & \\ \text{s.t.} & w_1y_1 & + & w_2y_2 \leq 50 & \forall w \text{ satisfying } w_1^2 + w_2^2 \leq 25. \end{array} \qquad (D_\infty)$$

   Show that $y = (10, 5)$ is not feasible in the true problem. Give a valid linear constraint that is violated by $y = (10, 5)$ and that could be added to (D_4).

(a) Get corresponding primal problem:

$$\begin{array}{lrcrcrcrcl} \min & 50x_1 & + & 50x_2 & + & 50x_3 & + & 50x_4 & & \\ \text{s.t.} & 5x_1 & & & + & 3x_3 & + & 2.5x_4 & = & 11 \\ & & & 5x_2 & + & 4x_3 & + & 4.33x_4 & = & 8 \\ & & & & & x_i \geq 0 & & & & \end{array} \qquad (P_4)$$

In (D_4), get dual slacks $s = (0, 25, 0, 3.35)$. So dual feasible.

So, need $x_2 = x_4 = 0$.

Then $\left.\begin{array}{rcl} 5x_1 + 3x_3 & = & 11 \\ 4x_3 & = & 8 \end{array}\right\} \Rightarrow \underbrace{x_1 = 1, \ x_3 = 2}_{\geq 0}$.

$x_1, x_3 \geq 0$

Thus, have dual feasibility, complementary slackness, and primal feasibility. So OPTIMAL.

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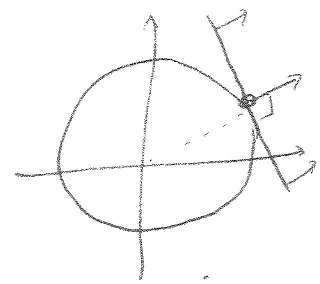
(b) To find violated constraint, solve:

$$\max \quad 10w_1 + 5w_2$$
$$\text{s.t.} \quad w_1^2 + w_2^2 \leq 25$$



Soln: $w = \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix}$,

with α chosen so that $\|w\| = 25$.

Thus, $\alpha = \sqrt{5}$.

~~Hence~~, Then $10w_1 + 5w_2 = 25\sqrt{5} > 50$ since $\sqrt{5} > 2$.

So $y = (10, 5)$ not feasible in (D_∞).

Constraint: $2\sqrt{5}\, y_1 + \sqrt{5}\, y_2 \leq 50$

Another violated constraint:

$$4y_1 + 3y_2 \leq 50$$

2. (30 points)
   Two systems of equations are:

$$\begin{array}{ll} (I) & Ad = 0,\ c^T d = 0,\ e^T d = 1,\ d \geq 0 \\ (II) & A^T v < tc,\ \cancel{t \geq 0},\ v \text{ free},\ t \text{ free} \end{array}$$

   where e is the n-vector of ones and t is a scalar.

   (a) (10 points) Show that exactly one of the two systems has a solution.

   (b) (10 points) Assume (P) and (D) are both feasible. Show that if (I) is consistent then (P) has an unbounded set of optimal solutions.

   (c) (10 points) Assume (D) is feasible. Show that if (II) is consistent then (D) has a strictly feasible solution.

(a) By Farkas, (I) has a solution $\Leftrightarrow$ the following system does not:

$$(III) \quad A^T w + \alpha c + \beta e \leq 0, \quad \beta > 0$$

(III) has a solution $\Leftrightarrow$ $A^T w < \gamma c$, γ, w free has a solution,

since $e = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ and need $\beta > 0$.

$\Leftrightarrow$ (II) has a solution

(b) Since both (P), (D) feasible, $\exists$ optimal primal soln x^*.
Let d solve (I). Then $x^* + \lambda d$ is feasible and optimal in (P) for any $\lambda \geq 0$

(c) Let $\hat{y}$ be feasible in (D), let v, t solve (II)
Rescale v, t if necessary so that $t > -1$.
Then let $\bar{y} = \hat{y} + v$, so $A^T \bar{y} = A^T \hat{y} + v < c + tc$
Let $\tilde{y} = \frac{1}{1+t} \bar{y}$, then $A^T \tilde{y} < c$, so $\tilde{y}$ is strictly feasible.

3. (30 points)
Let $(\hat{x}, \hat{y}, \hat{s})$ be strictly feasible solutions to (P) and (D). Let $\hat{\mu} = \hat{x}^T\hat{s}/n$. Assume the current iterate is reasonably well centered, so $\hat{X}\hat{S}e \approx \hat{\mu}e$, with the diagonal matrices $\hat{X}$ and $\hat{S}$ defined in the usual manner. Assume $\text{rank}(A) = m$.

Assume we add an extra variable w to (P), with corresponding column g, giving the modified problem

$$\begin{array}{lrcrcll} \min & c^Tx & + & hw & & & \\ \text{s.t.} & Ax & + & gw & = & c & \quad (Pw) \\ & & & x,\, w & \geq & 0 & \end{array}$$

where h is a scalar. The point $x = \hat{x}$, $w = 0$ is feasible for (Pw) but not strictly feasible. It is necessary to increase w to get a strictly feasible point.

(a) (15 points) Ignoring the additional dual constraint $g^Ty \leq h$, how would you try to find a strictly feasible solution to (Pw) while also keeping $XSe \approx \hat{\mu}e$? (Hint: set up conditions to be satisfied by primal directions $(\Delta x, \Delta w)$ and dual directions $(\Delta y, \Delta s)$.)

(b) (15 points) Now assume you've found a strictly feasible primal solution $(\bar{x}, \bar{w})$ and a new dual solution $\bar{y}$ with $\bar{s} = c - A^T\bar{y} > 0$. If you could pick the scalar h, how would you select it? (Aside: an algorithm could work from an incorrect h and gradually modify it to achieve the correct value.)

(a) Want to increase w, so ideally $\Delta w = 1$.

Then need $A\Delta x + g = 0$

Also, to keep $XSe \approx \hat{\mu}e$, require $\hat{S}\Delta x + \hat{X}\Delta s = \hat{\mu}e - \hat{X}\hat{S}e$.

Thus, solve

$$\begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ \hat{S} & 0 & \hat{X} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} 0 \\ -g \\ \hat{\mu}e - \hat{X}\hat{S}e \end{bmatrix}$$

(b) Let $\bar{\mu} = \dfrac{\bar{x}^T\bar{s}}{n}$. Would like extra complementarity term, namely $\bar{w}(\cancel{g}h - g^T\bar{y}) \approx \bar{\mu}$. So pick $h = \dfrac{\bar{\mu}}{\bar{w}} + g^T\bar{y}$.