Name:

MATP6640/DSES6770
Linear Programming
Spring 2006

Midterm Exam, Friday, April 7, 2006.

Please do all four problems. Show all work. No books or calculators allowed. You may use any result from class, the homeworks, or the texts, except where stated. You may use one sheet of handwritten notes. The exam lasts 110 minutes.

1. (30 points)
The following linear programming problem has an optimal solution x^* with x_1^*, x_2^*, and x_3^* basic:

$$\begin{array}{lrcrcrcrcrcl} \min & x_1 & + & 2x_2 & + & x_3 & + & x_4 & + & 5x_5 & & \\ \text{s.t.} & x_1 & + & x_2 & + & x_3 & + & 2x_4 & + & 2x_5 & = & b_1 \\ & & & x_2 & + & x_3 & & & + & 3x_5 & = & b_2 \\ & & & & & x_3 & + & x_4 & + & x_5 & = & b_3 \\ & & & & & & & & & x_i & \geq & 0 \quad i = 1, \ldots, 5 \end{array}$$

   (a) (7 points) What is the dual linear program?

   (b) (8 points) Use complementary slackness to solve the dual problem.

   (c) (7 points) For this part, assume $x_1^* > 0$, $x_2^* > 0$, and $x_3^* > 0$. Can you conclude that there are multiple optimal solutions to the primal problem?

   (d) (8 points) Assume b satisfies

$$-2b_1 + 2b_2 - b_3 = 0.$$

   Use duality to show that the optimal primal solution is degenerat.?

2. (15 points)
Consider the LP:

$$\begin{array}{lrcrcrcl} \min & 2x_1 & + & 6x_2 & + & 3x_3 & & \\ \text{s.t.} & x_1 & + & x_2 & + & x_3 & = & 6 \\ & x_1 & - & 2x_2 & + & 4x_3 & = & 9 \\ & & & & & x_i & \geq & 0 \quad i = 1, \ldots, 3 \end{array}$$

Show that $x = (3, 1, 2)$ is on the central trajectory for this problem.

3. (15 points)
Given n-vectors $x > 0$ and $s > 0$, show that the value of the scalar η that minimizes $||XSe - \eta e||^2$ is $\eta = \frac{x^T s}{n}$. Here X and S denote diagonal matrices containing the entries in x and s respectively, and e is the n-vector of ones.

4. (40 points, each part is worth 20 points)
The following is a primal-dual pair of linear programs:

$$\begin{array}{llcll} \min & c^T x & & & \\ \text{s.t.} & Ax & = & b & (P) \\ & Hx & = & h & \\ & x & \geq & 0 & \end{array} \qquad \begin{array}{llllll} \max & b^T y & + & h^T u & & \\ \text{s.t.} & A^T y & + & H^T u & \leq c & (D) \end{array}$$

Assume (P) and (D) are both feasible. Further, assume the polyhedron $Q := \{x \geq 0 : Ax = b\}$ is bounded. Let z denote the optimal value of (P) and (D).

For any appropriately dimensioned vector π, define

$$\Theta(\pi) := \min_x \{c^T x + \pi^T (h - Hx) : Ax = b, x \geq 0\}$$

This value $\Theta(\pi)$ provides an underestimate for z. The best lower bound is obtained by maximizing $\Theta(\pi)$.

(a) Show that the maximum value of $\Theta(\pi)$ is equal to z.

(b) The maximum value of $\Theta(\pi)$ can be overestimated by solving a linear programming problem of the form

$$\begin{array}{lcccll} \max & & & \theta & & \\ \text{s.t.} & F\pi & + & d\theta & \leq v & (LD) \end{array}$$

where F is a matrix and d and v are vectors. Assume an optimal solution $\bar{\pi}$, $\bar{\theta}$ for (LD) is known. Let $\bar{x}$ solve $\Theta(\bar{\pi})$, with corresponding dual multipliers $\bar{y}$. Assume $\bar{\theta} > \Theta(\bar{\pi})$, so $\bar{\theta}$ is still an overestimate. The bound $\bar{\theta}$ can be improved by adding a constraint to (LD). Give a valid linear constraint that must be satisfied by π and $\Theta(\pi)$ for all π.

Now assume all of the constraints in (LD) have the same form as the constraint you just found. What is the relationship between this approach to solving (P) and Dantzig-Wolfe Decomposition?