

# MATP6640/DSES6770 Linear Programming

## Midterm Exam, Spring 2004

Take Home

Due: Monday, March 29, 2004, in class.

This is to be all your own work. You may use any result from class, homeworks, the textbook, or the books on reserve in the library. Do not consult anybody or anything else. My email address is *mitchj@rpi.edu* and my phone number is 276-6915. I'll have my regular office hours on Thursday from 2pm to 4pm.

- (15 points.) Use linear programming duality to show that exactly one of the following alternatives holds.
  - There exists an  $x > 0$  with  $Ax = 0$ .
  - There exists a  $y$  with  $A^T y \leq 0$ ,  $A^T y \neq 0$ .
- (20 points; each part is worth 10 points.) Consider the linear program

$$\begin{array}{ll} \min & 2x_1 - 3x_2 + 4x_3 \\ \text{subject to} & x_1 - x_2 + 3x_3 = 2 \\ & x \in X \end{array}$$

where  $X = \{x : 0 \leq x_1 \leq 1, 0 \leq x_2, 0 \leq x_3 \leq 1\}$ .

- Solve the linear program using Dantzig-Wolfe decomposition. Initialize with the extreme points  $x^1 = (1, 0, 0)^T$  and  $x^2 = (0, 0, 1)^T$ .
  - Solve the linear program using the simplex method with the standard modification for handling upper bounds. That is, only include one constraint, and allow variables to be nonbasic at either their upper or lower bounds. Initialize with the basic feasible solution  $x = (0, 0, \frac{2}{3})$ .
- (40 points) We wish to partition a set  $S$  of  $n$  objects into  $K$  clusters, where  $K$  is given. There is a cost  $d_{ij} > 0$  associated with placing objects  $i$  and  $j$  in the same cluster. In one formulation of this problem, we use binary variables  $x_k$  corresponding to the nonempty subsets of the objects, with

$$x_k = \begin{cases} 1 & \text{if subset } k \text{ is one of the } K \text{ chosen clusters} \\ 0 & \text{otherwise.} \end{cases}$$

Let  $S^k$  be the  $k$ th nonempty subset of  $S$  for  $k = 1, \dots, 2^n - 1$ . Define

$$w_k := \sum_{i,j \in S^k, i < j} d_{ij}$$

to be the weight of  $S^k$ . We are going to examine the linear programming relaxation of this formulation, which we will denote  $(MP)$ , or Master Problem:

$$\begin{aligned} \min \quad & \sum_{k=1}^{2^n-1} w_k x_k \\ \text{subject to} \quad & Ax = e \\ & e^T x = K \\ & x \geq 0 \end{aligned} \quad (MP)$$

Here,  $e$  denotes a vector of ones of the appropriate dimension, and the  $k$ th column of  $A$  is the incidence vector of  $S^k$ . Note that  $A$  is an  $n \times (2^n - 1)$  matrix.

- (a) (15 points) Since there is an exponential number of subsets of the objects, we use a column generation approach. Thus, we work with a Revised Master Problem  $(RMP)$ , which only contains a subset of the variables. Show that we can determine whether the optimal solution to  $(RMP)$  is optimal in  $(MP)$  by solving a quadratic programming problem with integrality constraints.
- (b) Consider the problem with  $S := \{1, 2, 3, 4, 5\}$  and  $K = 3$ . The distances  $d_{ij}$  are given by the following matrix:

$$D := \begin{pmatrix} 0 & 8 & 21 & 17 & 18 \\ & 0 & 22 & 15 & 20 \\ & & 0 & 2 & 4 \\ & & & 0 & 3 \\ & & & & 0 \end{pmatrix}.$$

Initially, use the subsets  $\{1\}$ ,  $\{2\}$ ,  $\{3\}$ ,  $\{4\}$ ,  $\{5\}$ ,  $\{1, 2\}$ , and  $\{3, 4\}$  to provide the variables for  $(RMP)$ .

- i. (5 points) Show that there is only one feasible solution to  $(RMP)$ . Call it  $\bar{x}$ .
- ii. (10 points) The point  $\bar{x}$  is a degenerate BFS. Find a basis for which the reduced costs are all nonnegative.
- iii. (10 points) Show that  $\bar{x}$  does not solve  $(MP)$  by finding a column in  $(MP)$  with negative reduced cost. Add this column to  $(RMP)$  and solve using the revised simplex method.

4. (25 points; each part is worth 5 points.) Consider the primal-dual pair of linear programming problems:

$$\begin{array}{ll} \min & c^T x \\ \text{subject to} & Ax = b \quad (P) \\ & x \geq 0 \end{array} \qquad \begin{array}{ll} \max & b^T y \\ \text{subject to} & A^T y \leq c \quad (D) \end{array}$$

where  $x$  and  $c$  are  $n$ -vectors,  $b$  and  $y$  are  $m$ -vectors, and  $A$  is an  $m \times n$  matrix. Let  $\hat{b} = b - Ae$  and  $\hat{c} = c - e$ , where  $e$  denotes a vector of appropriate dimension with every component is equal to one. Let  $\hat{d} = c^T e + 1$ . Now consider the linear programming problem

$$\begin{array}{ll} \min & (n+1)w \\ \text{subject to} & Ax - bt + \hat{b}w = 0 \\ & -A^T y + ct - \hat{c}w \geq 0 \quad (HLP) \\ & b^T y - c^T x + \hat{d}w \geq 0 \\ & -\hat{b}^T y + \hat{c}^T x - \hat{d}t = -(n+1) \\ & x, t \geq 0, \quad y, w \text{ free,} \end{array}$$

where  $t$  and  $w$  are scalars.

- Find a feasible solution to  $(HLP)$  which satisfies all the inequalities strictly.
- Show that  $(HLP)$  is self-dual, that is, the dual to  $(HLP)$  is again the problem  $(HLP)$ .
- Show that the optimal value of  $(HLP)$  is zero.
- Suppose the optimal solution to  $(HLP)$  is  $(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{w})$ , with  $\tilde{t} > 0$  and  $\tilde{w} = 0$ . How would you use this solution to find optimal solutions to  $(P)$  and  $(D)$ ?
- An interior point method can be used to find a *strictly complementary* optimal solution to  $(HLP)$ . In such a solution, if the primal variable is equal to zero then the corresponding dual slack is strictly positive. Use this result to show that if  $t = 0$  in a strictly complementary optimal solution then either there exists a vector  $x \geq 0$  with  $Ax = 0$  and  $c^T x < 0$  or there exists a vector  $y$  with  $A^T y \leq 0$  and  $b^T y > 0$ . What can you conclude about  $(P)$  and  $(D)$ ?