MATP6640/DSES6770 Linear Programming

Midterm Exam, Spring 2002

Take Home Due: Thursday, March 28, 2002, in class.

This is to be all your own work. You may use any result from class, homeworks, the textbook, or the books on reserve in the library. Do not consult anybody or anything else. My email address is *mitchj@rpi.edu* and my phone number is 276–6915. I'll have my regular office hours on Wednesday from 10am to 12 noon.

In order that I can display grades, please write a 4 digit number on the front of your solution set.

1. (30 points)

   (a) (10 points) Consider the linear programming problem

   $$\begin{array}{lrcll} \min & c^T x & & & \\ \text{subject to} & Ax & = & b & (Pb) \\ & 0 \leq x & \leq & u & \end{array}$$

   where A is an $m \times n$ matrix, b is an m-vector, and c, u, and x are n-vectors. Notice that each variable x_i has a finite bound u_i. Show that the dual linear program to (Pb) is feasible.

   (b) (20 points) Consider the standard form primal-dual pair

   $$\begin{array}{lrcll} \min & c^T x & & & \\ \text{subject to} & Ax & = & b & (P) \\ & x & \geq & 0 & \end{array} \qquad \begin{array}{lrcccll} \max & b^T y & & & & & \\ \text{subject to} & A^T y & + & s & = & c & (D) \\ & & & s & \geq & 0 & \end{array}$$

   where A is an $m \times n$ matrix, b and y are m-vectors, and c, s, and x are n-vectors. Assume (P) and (D) are both feasible. Let Q_P and Q_D be the feasible regions for (P) and (D) respectively. Show that if x_i is bounded in Q_P then s_i is unbounded in Q_D.

2. (20 points) Let B be an $m \times m$ nonsingular matrix, and suppose $B = LQ$, where L is a lower-triangular $m \times m$ matrix and Q is an orthogonal $m \times m$ matrix (that is, $QQ^T = I$).

   (a) (5 points) Show that $BB^T = LL^T$.

   (b) (5 points) How would you solve the linear systems $B^T y = c$ and $Bd = a$ given L and Q. (The m-vectors c and a are known; the m-vectors y and d have to be found.)

(c) (5 points) Suppose the matrix Q is not stored and you are given only L. How can you solve $B^T y = c$ and $Bd = a$?

(d) (5 points) Suppose we now want to update the matrix B to the matrix $\bar{B}$ by replacing column p of B by the vector a. Denote column p of B by b and let e_p denote the pth unit vector. Then we can write

$$\bar{B} = B + (a - b)e_p^T.$$

Show that

$$\bar{L}\bar{L}^T = LL^T - bb^T + aa^T,$$

where $\bar{L}$ satisfies $\bar{B} = \bar{L}\bar{Q}$ for some orthogonal matrix $\bar{Q}$. (Note: You may assume that every $m \times m$ matrix M has a *QR-factorization*, where Q is orthogonal and R is uppertriangular, and $M = QR$. For this part, you do not need to derive $\bar{L}$ or $\bar{Q}$ explicitly.)

(e) (Extra credit: 5 points) How would you update L to find the lower triangular matrix $\bar{L}$?

3. (20 points) Let z^* be the finite optimal value of the feasible linear programming problem

$$\begin{array}{lrcll} \text{min} & c^T x & & & \\ \text{subject to} & Ax & = & b & (P) \\ & x & \geq & 0 & \end{array}$$

where A is an $m \times n$ matrix, b is an m-vector, and c and x are n-vectors. Let R_ϵ be the optimal value of the linear programming problem

$$\begin{array}{lrcll} \text{max} & e^T x & & & \\ \text{subject to} & Ax & = & b & (P_\epsilon) \\ & c^T x & \leq & z^* + \epsilon & \\ & x & \geq & 0 & \end{array}$$

where ϵ is a positive scalar and e is the n-vector of all ones. The quantity R_ϵ can be interpreted as the 1-norm of the largest vector in the ϵ level set $Q_\epsilon := \{x \geq 0 : Ax = b, c^T x \leq z^* + \epsilon\}$. Let $\epsilon' > \epsilon$. Show that

$$R_{\epsilon'} \leq \frac{\epsilon'}{\epsilon} R_\epsilon.$$

4. (10 points) Construct a four-vertex transshipment network with the following counterintuitive property: If the supply is decreased by one unit at an appropriately chosen vertex and demand is decreased by one unit at another appropriately chosen vertex then the optimal transshipment cost increases. (Both the original network flow problem and the modified problem should be feasible.)

5. (20 points) Use the network simplex algorithm to find the minimum cost flow for the problem with the following linear programming representation:

$$\begin{array}{lrcrcrcrcrcrcl}
\text{min} & & - & 2x_{13} & - & x_{23} & - & 2x_{24} & & & - & 10x_{41} & & \\
\text{subject to} & -x_{12} & - & x_{13} & & & & & & & + & x_{41} & = & 0 \\
& x_{12} & & & - & x_{23} & - & x_{24} & & & & & = & 0 \\
& & & x_{13} & + & x_{23} & & & - & x_{34} & & & = & 0 \\
& & & & & & & x_{24} & + & x_{34} & - & x_{41} & = & 0 \\
& \multicolumn{11}{r}{0 \le x_{12}, x_{13}, x_{23}, x_{24}, x_{34}} & \le & 1 \\
& \multicolumn{11}{r}{0 \le \quad x_{41}} & \le & 10
\end{array}$$

Use the initial basic feasible solution with basic variables:

$$x_{12} = 1, \quad x_{34} = 1, \quad x_{41} = 1$$

and nonbasic variables

$$x_{13} = 0, \quad x_{23} = 1, \quad x_{24} = 0.$$

You should need three iterations.