MATP6640/DSES6770 Linear Programming

Midterm Exam, Spring 2000

Take Home — Due: Tuesday, March 7, 2000, in class.

This is to be all your own work. You may use any result from class, homeworks, the textbook, or the books on reserve in the library. You may use the result of one question in a different question. Do not consult anybody or anything else. I can dispense hints to help you if you are stuck. My email address is *mitchj@rpi.edu* and my phone numbers are 276–6915(O) and 346–2811(H). I'll have my regular office hours on Wednesday from 12 noon to 2pm; I'll also have office hours next Monday, between 2pm and 4pm.

In order that I can display grades, please write a 4 digit number on the front of your solution set.

1. (20 points) A **Markov chain** has n different states and at each time period it may change its state. Given that it is in state i at time t, the probability that it is in state j at time $t+1$ is given by M_{ij}. The matrix M is the transition matrix for the chain. Notice that every entry $M_{ij} \geq 0$. Further, we must have $\sum_{j=1}^{n} M_{ij} = 1$. If p^t is the probability that the chain is in each different position at time t, then the probability distribution at time $t+1$ is given by $p^{t+1} = M^T p^t$. A **stationary distribution** p for this chain satisfies $M^T p = p$, $p \geq 0$, $e^T p = 1$, where e is the vector of ones. Use linear programming duality to show that every Markov chain has a stationary distribution. (Hint: Note that it suffices to show the existence of a nonzero $p \geq 0$ satisfying $M^T p = p$, since any such p can be scaled to satisfy $e^T p = 1$.)

2. (20 points) Construct a linear programming problem of the form

$$\begin{array}{lrcl} \min & c^T x & & \\ \text{s.t.} & Ax & \leq & b \\ & x & \geq & 0. \end{array} \qquad (P)$$

where A is a 4×2 matrix, x and c are 2-vectors and b is a 4-vector, such that the simplex algorithm takes 4 steps when started from an appropriate vertex with any rule for choosing the entering variable (provided it picks a variable with negative reduced cost).

3. (20 points) The attached sheet is part of a paper which was submitted to *Mathematical Programming*. The authors propose an algorithm for linear programming and then make some claims about the performance of this algorithm. Do these claims seem reasonable? Justify your answer.

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4. (40 points) We wish to partition a set S of n objects into K clusters, where K is given. There is a cost $d_{ij} > 0$ associated with placing objects i and j in the same cluster. In one formulation of this problem, we use binary variables x_k corresponding to the nonempty subsets of the objects, with

$$x_k = \begin{cases} 1 & \text{if subset k is one of the } K \text{ chosen clusters} \\ 0 & \text{otherwise.} \end{cases}$$

Let S^k be the kth nonempty subset of S for $k = 1, \ldots, 2^n - 1$. Define

$$w_k := \sum_{i,j \in S^k, i<j} d_{ij}$$

to be the weight of S^k. We are going to examine the linear programming relaxation of this formulation, which we will denote (MP), or Master Problem:

$$\begin{array}{llcl} \min & \sum_{k=1}^{2^n-1} w_k x_k & & \\ \text{subject to} & Ax & = & e \\ & e^T x & = & K \\ & x & \geq & 0 \end{array} \qquad (MP)$$

Here, e denotes a vector of ones of the appropriate dimension, and the kth column of A is the incidence vector of S^k. Note that A is an $n \times (2^n - 1)$ matrix.

(a) (15 points) Since there is an exponential number of subsets of the objects, we use a column generation approach. Thus, we work with a Revised Master Problem (RMP), which only contains a subset of the variables. Show that we can determine whether the optimal solution to (RMP) is optimal in (MP) by solving a quadratic programming problem with integrality constraints.

(b) Consider the problem with $S := \{1, 2, 3, 4, 5\}$ and $K = 3$. The distances d_{ij} are given by the following matrix:

$$D := \begin{pmatrix} 0 & 8 & 21 & 17 & 18 \\ & 0 & 22 & 15 & 20 \\ & & 0 & 2 & 4 \\ & & & 0 & 3 \\ & & & & 0 \end{pmatrix}.$$

Initially, use the subsets $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{1, 2\}$, and $\{3, 4\}$ to provide the variables for (RMP).

i. (5 points) Show that there is only one feasible solution to (RMP). Call it $\bar{x}$.

ii. (10 points) The point $\bar{x}$ is a degenerate BFS. Find a basis for which the reduced costs are all nonnegative.

iii. (10 points) Show that $\bar{x}$ does not solve (MP) by finding a column in (MP) with negative reduced cost. Add this column to (RMP) and make one pivot using the revised simplex method. Is the new iterate optimal in (RMP)?

iv. (Extra credit: up to 10 points) Is the new iterate optimal in (MP)?