1. $\min \; \|Xs - \mu e\|^2$

$\text{s.t.} \; A^T y + s = c$

KKT conditions:

Let π denote KKT multipliers

Derivative wrt y: $\quad A\pi = 0 \qquad (1)$

Derivative wrt s: $\quad 2X(Xs - \mu e) + \pi = 0 \qquad (2)$

Constraint: $\quad A^T y + s = c \qquad (3)$

Thus: $\frac{1}{2}(2) - X^{+2}(3)$:

$$-\mu Xe + \frac{1}{2}\pi - X^2A^Ty = -X^2c \qquad (4)$$

Then $\frac{1}{2}(1) - A(4) \Rightarrow$

$$\mu AXe + AX^2A^Ty = AX^2c$$

So $\quad y^* = (AX^2A^T)^{-1}(Ac - \mu AXe) = (AX^2A^T)^{-1}(AX^2c - \mu Ax)$

$$= \underline{(AX^2A^T)^{-1}(AX^2c - \mu b)}$$

Then $\quad s^* = c - A^Ty^* = \underline{c - A^T(AX^2A^T)^{-1}AX^2c + \mu A^T(AX^2A^T)^{-1}b}$

$$\pi = 2\mu Xe - 2X^2s$$

$$= 2\mu x - 2X^2c + 2X^2A^T(AX^2A^T)^{-1}AX^2c - 2\mu X^2A^T(AX^2A^T)^{-1}b$$

$$= -2XP_{AX}Xc + 2\mu XP_{AX}e$$

with $\quad P_{AX} = I - XA^T(AX^2A^T)^{-1}AX$

$$\|Xs-\mu e\|^2 = \sum_{i=1}^{n} (x_i s_i - \mu)^2 = v$$

If $v < \mu^2$ then $x_i s_i > 0 \ \forall i$, else $(x_i s_i - \mu)^2 > \mu^2$.
Since $x > 0$, we immediately get $s_i^* > 0 \ \forall i$

The dual to the original has a strictly feasible solution.
Thus, the set of optimal primal solutions is bounded.

Alternative derivation of y^*, s^*:

$$\|Xs-\mu e\|^2 = \|X(c-A^Ty)-\mu e\| =: f(y)$$

Solve $Df(y) = 0$:

$$0 = Df(y) = 2AX\left(X(c-A^Ty) - \mu e\right) = 2AX^2c - 2\mu Ax - 2AX^2A^Ty$$

$$= 2AX^2c - 2\mu b - 2(\underbrace{AX^2A^T}_{\text{invertible}})y$$

So $y^* = (AX^2A^T)^{-1}(AX^2c - \cancel{2}\mu b)$

and $s^* = c - A^Ty^*$.

2. $\max\ 17y_1 + y_2$

s.t. $y_1 + 3y_2 + s_1 = 3$

$y_1 - y_2 + s_2 = 3$

$y_1 + y_2 + s_3 = 2$

$s_i \geq 0,\ i = 1, 2, 3.$

$y = (1, -1) \Rightarrow$

$s = (5, 1, 2)$

Dual is: $\min\ 3x_1 + 3x_2 + 2x_3$

s.t. $x_1 + x_2 + x_3 = 17$

$3x_1 - x_2 + x_3 = 1$

$x_i \geq 0,\ i = 1, \ldots, 3.$

Central path: Need $Ax = b$, $x > 0$, $x_i s_i = \mu$ for some μ.

So: $x_1 = \frac{1}{5}\mu, \quad x_2 = \mu, \quad x_3 = \frac{1}{2}\mu$

Thus, need μ to satisfy:

$\frac{1}{5}\mu + \mu + \frac{1}{2}\mu = 17 \Rightarrow 2\mu + 10\mu + 5\mu = 170$
$\Rightarrow \mu = 10$

$\frac{3}{5}\mu - \mu + \frac{1}{2}\mu = 1$
$\Rightarrow 6\mu - 10\mu + 5\mu = 10$
$\Rightarrow \mu = 10$

So $\mu = 10$, $x = (2, 10, 5)$ works.

3. (a) Eg: $\min \ 2x_1 - x_2$

$$\text{s.t.} \quad x_1 + 2x_2 = 0$$
$$x_i \geq 0.$$

Only optimal soln is $x = (0,0)$, so $\mathcal{B} = \phi$

(b) Eg: $\min \ 3x_1 + 2x_2 + x_3$

$$\text{s.t.} \quad 3x_1 + 2x_2 + x_3 = 6.$$
$$x_i \geq 0 \quad \forall i.$$

Optimal solutions include $(1,1,1)$.

So $\mathcal{B} = \{1, 2, 3\}$.