1. Let x^* solve the stochastic program so

$$v^* = \min_{x \in X} f(x).$$

Then $\hat{f}(x^*) \geq \hat{v}_k^*$, since x^* is feasible in the k-scenario problem.

So,
$$\begin{aligned} E_\xi(\hat{v}_k^*) &\leq E_\xi(\hat{f}(x^*)) \\ &= \frac{1}{k}\sum_{i=1}^{k} E_\xi(g(x^*, \xi_i)) \\ &= \frac{1}{k}\sum_{i=1}^{k} f(x^*) \\ &= f(x^*) \\ &= v^*. \end{aligned}$$

2. $E(\bar{f}_l(x)) = \frac{1}{l} \sum_{i=1}^{l} E(g(x, \xi_i))$

$$= \frac{1}{l} \sum_{i=1}^{l} f(x)$$

$$= f(x)$$

$$\geq v^*$$

3. When using these three scenarios, the optimal value of the LP is

$$-0.062666$$

Solution is $\hat{x}^* = (0.2, 0.35, 0.35, 0.1)$.

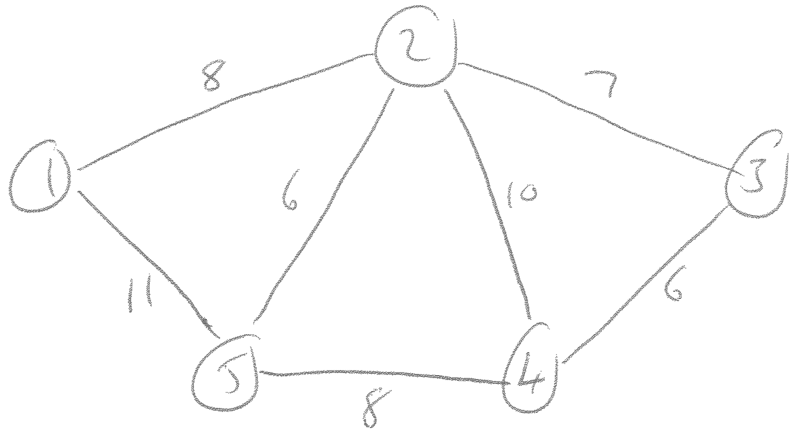
4. Plugging in $\hat{x}^*$ from Q3 gives a value of

$$\bar{F}_L(\hat{x}^*) = -0.0714$$

Optimal solution is $\hat{x}^* = (0.35, 0.35, 0.2, 0.1)$,

with value -0.07305.

5.



$$\min \sum_{p\in P} c_p f_p$$

$$\text{st.} \quad \sum_{p\in P_k} f_p = b_k \quad \text{for each commodity } k$$

$$\sum_{p\in P:\, e\in p} f_p \le u_e \quad \text{for each edge } e\in E.$$

$$f_p \ge 0 \quad \forall p\in P$$

With the four paths: Path A1 : 1 – 2 – 3 $\quad c_{A1} = 15$

A2 : 1 – 5 – 4 – 3 $\quad c_{A2} = 25$

B1 : 5 – 2 – 3 $\quad c_{B1} = 13$

~~B2~~ C1: 4 – 5 – 1 $\quad c_{C1} = 19$

Q5 p2

LP: $\min\ 15 f_{A1} + 25 f_{A2} + 13 f_{B1} + 19 f_{C1}$

$$
\begin{array}{llr}
\text{s.t.} & f_{A1} + f_{A2} = 25 & A \\
& f_{B1} = 20 & B \\
& f_{C1} = 20 & C \\
& f_{A1} \le 25 & (1,2) \\
& f_{A2} + f_{C1} \le 40 & (1,5) \\
& f_{A1} + f_{B1} \le 35 & (2,3) \\
& 0 \le 50 & (2,4) \\
& f_{B1} \le 40 & (2,5) \\
& f_{A2} \le 25 & (3,4) \\
& f_{A2} + f_{C1} \le 50 & (4,5) \\
& f_i \ge 0 \quad \forall i &
\end{array}
$$

Dual:

$$\max\ 25 y_A + 20 y_B + 20 y_C - 25\pi_{12} - 40\pi_{15} - 35\pi_{23} - 50\pi_{24} - 40\pi_{25} - 25\pi_{34} - 50\pi_{45}$$

$$
\begin{array}{ll}
\text{s.t.} & y_A - \pi_{12} - \pi_{23} \le 15 \\
& y_A - \pi_{15} - \pi_{34} - \pi_{45} \le 25 \\
& y_B - \pi_{23} - \pi_{25} \le 13 \\
& y_C - \pi_{15} - \pi_{45} \le 19 \\
& \pi_{ij} \ge 0 \quad \forall (i,j).
\end{array}
$$

Solutions: $f_{A1} = 15$, $f_{A2} = 10$, $f_{B1} = 20$, $f_{C1} = 20$

From complementary slackness:

$\pi_{15} = \pi_{12} = \pi_{24} = \pi_{25} = \pi_{34} = \pi_{45} = 0$

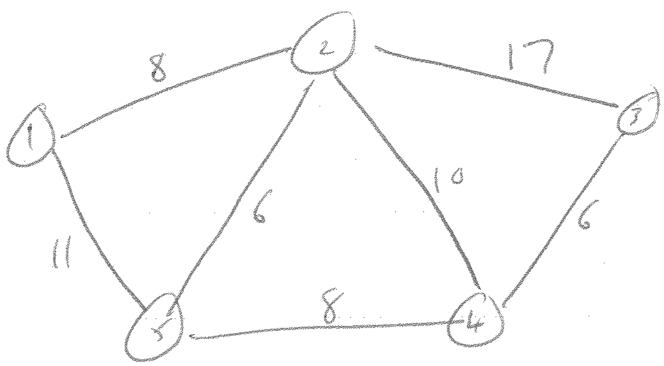
$\cancel{y_B = 13}$, $y_C = 19$. $y_A = 25$, $\cancel{\pi_{12} = \alpha}$, $\cancel{0 \le \alpha \le 0}$

~~Choose α = 0~~, so $\cancel{y_A = 15}$, $\cancel{\pi_{12} = 0}$. $\pi_{23} = 10$, $y_B = 23$

Reduced cost for a path p for commodity k

$= $ dual slack for a path p for commodity k

$$= \sum_{e \in p} c_e + \sum_{e \in p} \pi_e - y_k$$



Solve shortest path problem for each commodity, with modified costs:

Commodity A: 1 – 2 – 4 – 3, cost $24 < y_A = 25$

Commodity B: 5 – 4 – 3, cost $14 < y_B = 23$

Commodity C: 4 – 2 – 1, cost $18 < y_C = 19$.

All three paths have negative reduced cost.

Add path B2: 5 – 4 – 3 to the model.

$$\min \quad 15 f_{A1} + 25 f_{A2} + 13 f_{B1} + 14 f_{B2} + 19 f_{C1}$$

$$
\begin{aligned}
\text{st} \quad & f_{A1} + f_{A2} &= 25 && A \\
& f_{B1} + f_{B2} &= 20 && B \\
& f_{C1} &= 20 && C \\
& f_{A1} &\le 25 && (1,2) \\
& f_{A2} + f_{C1} &\le 40 && (1,5) \\
& f_{A1} + f_{B1} &\le 35 && (2,3) \\
& 0 &\le 50 && (2,4) \\
& f_{B1} &\le 40 && (2,5) \\
& f_{A2} + f_{B2} &\le 25 && (3,4) \\
& f_{A2} + f_{B2} + f_{C1} &\le 50 && (4,5) \\
& f_i \ge 0 \quad \forall i.
\end{aligned}
$$

Dual:

$$\max \quad 25 y_A + 20 y_B + 20 y_C - 25\pi_{12} - 40\pi_{15} - 35\pi_{23} - 50\pi_{24} - 40\pi_{25} - 25\pi_{34} - 50\pi_{45}$$

$$
\begin{aligned}
\text{st} \quad & y_A - \pi_{12} - \pi_{23} &\le 15 && A1 \\
& y_A - \pi_{15} - \pi_{34} - \pi_{45} &\le 25 && A2 \\
& y_B - \pi_{23} - \pi_{25} &\le 13 && B1 \\
& y_B - \pi_{34} - \pi_{45} &\le 14 && B2 \\
& y_C - \pi_{15} - \pi_{45} &\le 19 && C1 \\
& \pi_{ij} \ge 0 \quad \forall (i,j) \in E
\end{aligned}
$$

Soln: $f_{C1} = 20, \; f_{A1} = 25, \; f_{A2} = 0, \; f_{B1} = 10, \; f_{B2} = 10$

By complementary slackness:

$\pi_{15} = \pi_{24} = \pi_{25} = \pi_{34} = \pi_{45} = 0$

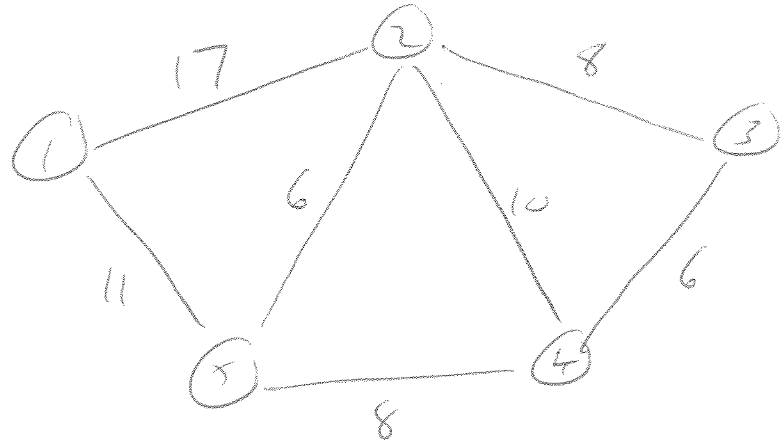
$\cancel{y_A = }$

$y_B = 14, \quad y_C = 19, \quad \pi_{23} = 1$

Choose $\pi_{12} = 9, \; y_A = 25$ (have multiple optimal dual solutions)

Q5, p5

Solve shortest path problem with modified costs $c+\pi$:



Commodity A: Shortest path: 1-2-3, cost 25 $= y_A$ ✓

Commodity B: Shortest path: 5-2-3, cost 14 $= y_B$ ✓

Commodity C: Shortest path: 4-5-1, cost 19 $= y_C$ ✓

So all reduced costs nonnegative, so OPTIMAL.