1. Let x^* solve (P), y^*, u^* solve (D).

Then
$$\theta(u^*) = \min_{x} \; c^T x + u^{*T}(h - Hx)$$
$$\text{s.t. } Ax = b, \; x \geq 0$$

$$= u^{*T}h + \max \; b^T y$$
$$\text{s.t. } A^T y \leq c - H^T u^*$$

$$\geq u^{*T}h + b^T y^*, \text{ since } y^* \text{ feasible here}$$

$$= c^T x^*.$$

We're told $\theta(u) \leq c^T x$ for any feasible x, u.

So $\theta(u^*)$ must achieve the maximum for $\theta(\pi)$,

and $\theta(u^*) = c^T x^* = z$.

2. $$\partial(\pi) = \min_{x} \; c^T x + \pi^T (h - Hx)$$
$$\text{s.t.} \quad Ax = b$$
$$x \geq 0$$

$$\leq \; c^T \bar{x} + \pi^T (h - H\bar{x})$$ since $\bar{x}$ feasible in here

This is the valid linear constraint.

It holds at equality at $\pi = \bar{\pi}$.

3. Then (LD) has the form:

$$\max \quad \theta \qquad\qquad (LD)$$

$$\text{s.t.} \; -(h - Hx^k)^T \pi + \theta \le c^T x^k \quad \text{for extreme points } x^k.$$

Taking the dual, we obtain

$$\min \sum_k \lambda_k c^T x^k$$

$$\begin{aligned} \text{s.t.} \; -\sum_k \lambda_k (h - Hx^k) &= 0 \\ \sum_k \lambda_k &= 1 \\ \lambda_k &\ge 0 \quad \forall k \end{aligned}$$

Rewriting the first constraint, and exploiting $\sum \lambda_k = 1$, we obtain:

$$\min \sum_k \lambda_k c^T x^k$$

$$\begin{aligned} \text{s.t.} \quad \sum_k \lambda_k H x^k &= h \\ \sum_k \lambda_k &= 1 \\ \lambda_k &\ge 0 \quad \forall k. \end{aligned}$$

This is exactly the Dantzig-Wolfe master problem.

The D-W subproblem is exactly the LP given by $\theta(\pi)$.

4.

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & Hx = h \\ & x \geq 0 \end{aligned} \qquad (P) \qquad\qquad \begin{aligned} \max \quad & b^T y + h^T w \\ \text{s.t.} \quad & A^T y + H^T w \leq c \end{aligned} \qquad (D)$$

Subproblem, and its dual:

$$\begin{aligned} \min \quad & (c - A^T \bar{\pi})^T x \\ \text{s.t.} \quad & Hx = h \\ & x \geq 0 \end{aligned} \qquad (SP) \qquad\qquad \begin{aligned} \max \quad & h^T v \\ \text{s.t.} \quad & H^T v \leq c - A^T \bar{\pi} \end{aligned} \qquad (SD)$$

Let $\bar{v}$ be optimal in (SD). Then $s^* = h^T \bar{v}$.

Also, $y = \bar{\pi}$, $w = \bar{v}$ is feasible in (D).

So $c^T x \geq b^T \bar{\pi} + h^T \bar{v} = s^* + b^T \bar{\pi}$ for any feasible x.

This gives a lower bound on the optimal value of (P).

It also gives a possible termination criterion:

stop if duality gap small enough.

5. Look at the dual pair:

$$\begin{array}{llll} \min & b^T y & & \\ \text{s.t.} & A^T y = 0 & (P) & \\ & y \geq 0 & & \end{array} \qquad \begin{array}{lll} \max & 0 & \\ \text{s.t.} & Ax \leq b & (D) \end{array}$$

① Assume $b^T y \geq 0$ is redundant.

Then (P) has optimal value 0 (since 0 is feasible)

So (D) has optimal value 0.

So $Ax \leq b$ is consistent.

② Assume $b^T y \geq 0$ is not redundant.

So $\exists \bar{y}$ with $A^T \bar{y} = 0$, $\bar{y} \geq 0$, but $b^T \bar{y} < 0$.

Then (P) is unbounded below (consider $\alpha \bar{y}$, let $\alpha \to +\infty$)

So (D) is infeasible

So $Ax \leq b$ is inconsistent.