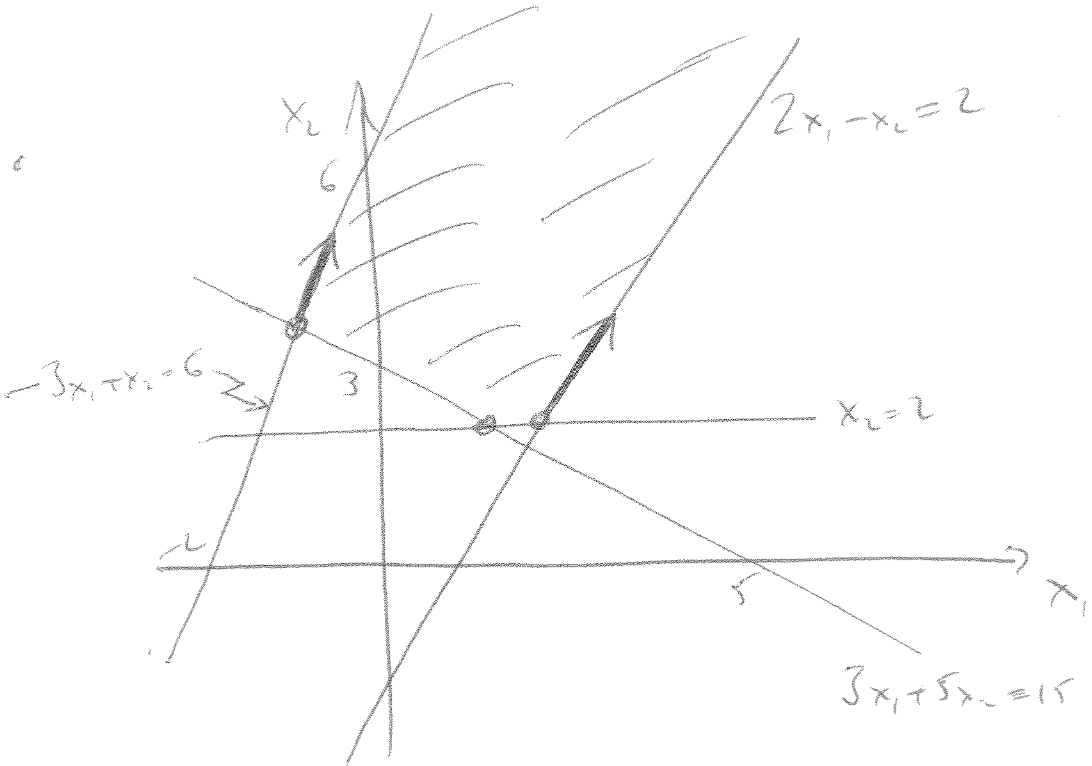
1.



Extreme points: $\begin{bmatrix} -5/6 \\ 3\frac{1}{2} \end{bmatrix}$, $\begin{bmatrix} 5/3 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

Extreme rays: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} -5/6 & 1\frac{2}{3} & 2 \\ 3\frac{1}{2} & 2 & 2 \end{bmatrix}$.

2. $\mathbb{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ (Basic variables could be ordered differently.)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}}_{L}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

$\mathbb{B}^T y = c_{\mathbb{B}}$, so $U^T z = c_{\mathbb{B}}$ and $y = L^T z$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} z = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow z = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow y = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Check: $y^T \mathbb{B} = [1 \;\; 1 \;\; 1]$ ✓

Reduced coeffs: $c_N^T - y^T N = [-3 \;\; 1] - [0 \;\; 1 \;\; 1]\begin{bmatrix} 0 & -2 \\ -3 & 2 \\ 1 & -3 \end{bmatrix} = [-1 \;\; 2]$

So x_2 enters the basis

$$B\,d = a_2, \quad \text{so} \quad U d_B = L a_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ -2 \end{bmatrix}$$

$$\text{Thus, } \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ -2 \end{bmatrix} \Rightarrow d_B = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \begin{matrix} x_1 \\ x_3 \\ x_4. \end{matrix}$$

So x_2 replaces x_1 in basis.

Value of x_2 is $6/1 = 6$.

$$\text{Update } \begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix} \longrightarrow \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix} - 6 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 15 \end{bmatrix}.$$

Check: $Ax = b$ ✓

$$\text{Elementary matrix: } E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix},$$

New basis matrix $B_1 = B E_1$,

so $L B_1 = U E_1$

$B_1^T y = c_B$, so $E_1^T w_1 = c_B$ and $U^T z = w_1$ and $y = L^T z$:

$$\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \\ w \\ \\ \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \Rightarrow w = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} z = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \Rightarrow z = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Check: $B^T y = c_B$ ✓

Reduced costs: $c_N^T - y^T N = \begin{bmatrix} 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 2 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix}$

So x_5 enters the basis.

$Bd = a_5$, so $Up = La_5$, $E_1 d = p$:

$$La_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} p = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \Rightarrow p = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} d = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \Rightarrow d = \begin{array}{c} \begin{bmatrix} -3 \\ -2 \\ -5 \end{bmatrix} \end{array} \begin{array}{l} x_2 \\ x_3 \\ x_4. \end{array}$$

Since $d \leq 0$, the problem is unbounded.

Ray is

$$r = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 5 \\ 1 \end{bmatrix}$$

negative of components of d for basic variables

for incoming nonbasic variable

Check: $Ar = 0$ ✓

$c^T r = -1 < 0$ ✓.

4. Eg:

$$
\begin{aligned}
\min\ & x_1 + x_2 \\
\text{s.t.}\ & x_1 + x_2 = 0 \qquad (P) \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Unique optimal solution: $x = 0$

Dual:

$$
\begin{aligned}
\max\ & 0 \\
\text{s.t.}\ & y \leq 1 \\
& y \leq 1 \\
& y
\end{aligned}
$$

Optimal soln is $y = 1$, with two components of s equal to zero.

A less trivial example:

$$
\begin{aligned}
\min\ & \qquad\qquad\qquad x_4 \\
\text{s.t.}\ & x_1 \qquad\ + x_3 - x_4 = 0 \qquad (P) \\
& \qquad x_2 - x_3 + 2x_4 = 2 \\
& x_i \geq 0
\end{aligned}
$$

Unique primal optimal: $x = (0, 2, 0, 0)$, value $= 0$

Dual:

$$
\begin{aligned}
\max\ & 2y_2 \\
\text{s.t.}\ & y_1 \leq 0 \\
& y_2 \leq 0 \\
& y_1 - y_2 \leq 0 \\
& -y_1 + 2y_2 \leq 1
\end{aligned}
$$

Optimal: $y = (0, 0)$, value $= 0$,
dual slacks $= (0, 0, 0, 1)$.

Any example must be primal degenerate.

5. Dual:

$$
\begin{array}{lrcl|c}
 & & & & \text{Ray } r \\
\max & 10y_1 + 7y_2 + 6y_3 & & & \\
\text{s.t.} & y_1 \qquad\quad + y_3 & \le & 1 & 0 \\
 & -3y_2 + y_3 & \le & -3 & 3 \\
 & y_1 + y_2 \qquad & \le & 1 & 2 \\
 & y_2 \qquad & \le & 1 & 5 \\
 & -2y_1 + 2y_2 - 3y_3 & \le & 1 & 1
\end{array}
$$

Adding using components of r:

Any dual solution must satisfy:

$$0y_1 + 0y_2 + 0y_3 \le -1,$$

that is, $0 \le -1.$

Clearly infeasible.