1. Let $\bar{x}$ be an optimal solution that is not a BFS.

Then the columns used by $\bar{x}$ must be linearly dependent.

So there exists d satisfying $Ad = 0$, $d \neq 0$,
$d_j = 0$ if $\bar{x}_j = 0$ $\forall j$.

$\bar{x} \pm \varepsilon d$ is feasible for sufficiently small positive ε.

Since $\bar{x}$ is optimal, we must have $c^T d = 0$, else either $\bar{x} + \varepsilon d$ or $\bar{x} - \varepsilon d$ is better than $\bar{x}$.

Assume WLOG that d has at least one positive component (else, let $d = -d$).

Let $\bar{\varepsilon} = \min \left\{ \frac{\bar{x}_j}{d_j} : d_j > 0 \right\}$.

Then $\bar{x} - \bar{\varepsilon} d$ is optimal and has one less positive component than $\bar{x}$.

Update $\bar{x} \leftarrow \bar{x} - \bar{\varepsilon} d$.

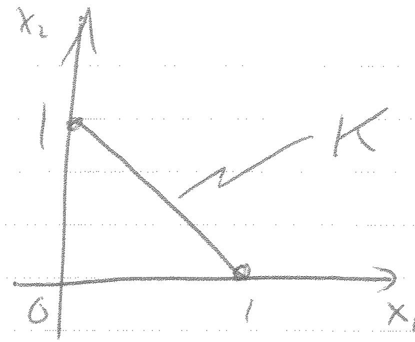
If $\bar{x}$ is BFS, done, else repeat the process.

2. (a) $\min \; x_1 + 3x_2$

s.t. $x_1 + x_2 = 1$

$2x_1 + 2x_2 = 2$

$x_i \geq 0 \; \forall i$



$n = 2, \; m = 2, \quad \dim(K) = 1$

(b) $\min \; x_1 + 2x_2 + 3x_3$

s.t. $x_1 + x_2 + x_3 = 1$

$x_1 = 1$

$x_i \geq 0 \quad \forall i$

$n = 3, \; m = 2, \quad K = \{(1,0,0)\}$, so $\dim(K) = 0$.

(c) Degenerate BFS $\bar{x} = (1, 0, 0)$.

Bases: (i) Columns 1 and 2.

(ii) Columns 1 and 3.

3a. $\max \quad 6y_1 - 5y_2 + 8y_3$

$$\text{s.t.} \quad \begin{array}{rcrcrcr} y_1 & & & + & y_3 & = & 3 \\ & & 2y_2 & + & y_3 & \geq & -5 \\ -y_1 & - & y_2 & & & \leq & 8 \end{array}$$

$$y_1 \text{ free}, \; y_2 \geq 0, \; y_3 \leq 0$$

By complementary slackness, if $\bar{x} = (6,0,0)$, need:

$$y_2 = y_3 = 0, \quad \text{since} \quad 2y_2 + y_3 > -5$$
$$\text{and} \quad -y_1 - y_2 < 8$$

So $\quad y = (3, 0, 0).$

This is dual feasible, with value 18, equal to primal value.

Primal equivalent to: $\quad (x_1 \longrightarrow x_1 - x_4 \,,\; x_2 \longrightarrow -x_5)$

$$\min \; 3x_1 - 3x_4 + 5x_5 + 8x_3$$

$$\text{s.t.} \quad \begin{array}{rrrrrrl} x_1 - x_4 & & - x_3 & & & = & 6 \\ & -2x_5 & - x_3 & - x_6 & & = & -5 \\ x_1 - x_4 & - x_5 & & & + x_7 & = & 8 \end{array}$$

$$x_i \geq 0, \quad i = 1, \ldots, 7.$$

4. Construct the Primal-Dual pair:

$$\begin{array}{llcll} \min & c^T x & (P) & \max & 0 \quad (D) \\ \text{s.t.} & Ax = 0 & & \text{s.t.} & A^T y = c \end{array}$$

(P) is always feasible: take $x = 0$, with value 0.

So two cases:

(i) (P) is unbounded $\Rightarrow$ (D) is infeasible $\Rightarrow$ (b) inconsistent

$\Downarrow$

(a) consistent

So exactly one of (a), (b) holds

(ii) (P) has finite optimal value $\Rightarrow$ (D) feasible $\Rightarrow$ (b) consistent

$\Downarrow$

(D) has optimal value 0

$\Rightarrow$ (P) has optimal value 0

$\Rightarrow$ (a) is inconsistent.

(Note: if $\exists \bar{x}$ with $c^T \bar{x} > 0$ and $A\bar{x} = 0$, then $-\bar{x}$ satisfies $c^T x < 0$, $Ax = 0$, so (P) is unbounded.)