GAME THEORY (Chvátal, Chapter 15)

2-person Zero-Sum Games

Each player has a set of options available.

Given Player 1's and Player 2's choices there is a PAYOFF of a from 1 to 2 or from 2 to 1

Eg: Rock / Paper / Scissors

		2: Rock	Paper	Scissors
	Rock	0	-1	1
1	Paper	1	0	-1
	Scissors	-1	1	0

← Net Payoff to Player 1

Eg: Each player hides either a nickel or a dime.

If the coins agree, Player 1 gets both
If the coins disagree, Player 2 gets both

		2: Nickel	Dime
1	Nickel	10	-15
	Dime	-15	20

Really,

$$A = \begin{bmatrix} 5 & -5 \\ \cancel{-5}\ -10 & 10 \end{bmatrix}$$

Value $= 0$, $x = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix}$, $y = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

$\cancel{y = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix}}$.

GAME 2.

Let A denote the payoff matrix.

Let x denote Player 1's strategy.

Have m choices. $x_i \overset{\text{prob}}{=}$ choose i ~~with prob x_i~~

So $\sum_{i=1}^{m} x_i = 1$, $x_i \geq 0$ $i=1,\dots,m$

Let y denote Player 2's strategy

Have n choices. y_j = prob choose j.

~~Eg Rock / Paper / Scissors~~

If Player 2 has a strategy y, what should Player 1's strategy be?

Expected payoff to player 1 is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij} y_j = x^T A y$$

So: should try:

$$\max_x \; x^T A y$$
$$\text{s.t.} \quad e^T x = 1$$
$$x \geq 0$$

This is a continuous knapsack problem, and a bfs has one basic variable.

So if y is fixed and known, Player 1 uses a pure strategy,

picking $x_i = 1$ for the i with the largest value of $(Ay)_i$.

Eg: Rock / Paper / Scissors:

With $y_{Rock} = \frac{1}{2}$, $y_{Paper} = \frac{1}{4}$, $y_{Scissors} = \frac{1}{4}$.

Then $Ay = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{4} \\ -\frac{1}{4} \end{bmatrix}$

So pick $x_{Paper} = 1$, $x_{Rock} = x_{Scissors} = 0$. (so $x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.)

Expected payoff to Player 1 is $\frac{1}{4} (= x^T A y)$

In particular:

1	2	Prob	Payoff
Paper	Rock	$\frac{1}{2}$	1
Paper	Paper	$\frac{1}{4}$	0
Paper	Scissors	$\frac{1}{4}$	-1

Expected payoff: $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

Player 2 wants to make the expected payoff to Player 1 as small as possible.

So Player 2 solves:

$$\min_{\substack{y \geq 0 \\ e^T y = 1}} \left\{ \max_{\substack{x \geq 0 \\ e^T x = 1}} x^T A y \right\}$$

Eg: if $y = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$ then $\max_{\substack{x \geq 0 \\ e^T x = 1}} x^T A y$ is 0, achieved

by all feasible x, since $Ay = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

If Player 1 has to consider different Player 2 strategies:

Player 1 solves:

$$\max_{\substack{x \geq 0 \\ e^T x = 1}} \left\{ \min_{\substack{y \geq 0 \\ e^T y = 1}} x^T A y \right\}$$

if Player 2 knows x, Player 2 would choose the best y for this problem.

<u>Thm</u> (Von Neumann)
$$\max_{\substack{x \geq 0 \\ e^T x = 1}} \left\{ \min_{\substack{y \geq 0 \\ e^T y = 1}} x^T A y \right\} = \min_{\substack{y \geq 0 \\ e^T y = 1}} \left\{ \max_{\substack{x \geq 0 \\ e^T x = 1}} x^T A y \right\}$$

<u>Proof</u> Look at the inner problem on the LHS:

for a given x,
$$\min_{\substack{y \geq 0 \\ e^T y = 1}} x^T A y \equiv \min_{\substack{y \geq 0 \\ e^T y = 1}} \sum_{j=1}^{n} \left(\sum_{i=1}^{m} x_i a_{ij} \right) y_j$$

An optimal solution is a ~~y~~ pure strategy, namely,

$$\cancel{y_j = \begin{cases} 1 & \text{if } \sum_{i=1}^{m} x_i a_{ij} = \min_k \sum_{i=1}^{m} x_i a_{ik} \\ 0 & \text{otherwise} \end{cases}}$$

let $k = \arg\min_j \left\{ \sum_{i=1}^{m} a_{ij} x_i \right\}$,

and let $y_k = 1$, $y_j = 0$ otherwise

So
$$\max_{\substack{x \geq 0 \\ e^T x = 1}} \left\{ \min_{\substack{y \geq 0 \\ e^T y = 1}} x^T A y \right\}$$

$$= \max_{\substack{x \geq 0 \\ e^T x = 1}} \min_{j} \left\{ \sum_{i=1}^{m} x_i a_{ij} \right\}$$

$$\begin{aligned} = \max \quad & z \\ \text{s.t.} \quad & z - \sum_{i=1}^{m} x_i a_{ij} \leq 0 \qquad j = 1, \ldots, n \\ & e^T x = 1 \\ & x \geq 0 \end{aligned} \qquad (1)$$

Similarly,
$$\min_{\substack{y \geq 0 \\ e^T y = 1}} \left\{ \max_{\substack{x \geq 0 \\ e^T x = 1}} x^T A y \right\}$$

$$= \min_{\substack{y \geq 0 \\ e^T y = 1}} \max_{i} \sum_{j=1}^{n} a_{ij} y_j$$

$$\begin{aligned} = \min \quad & w \\ \text{s.t.} \quad & w - \sum_{j=1}^{n} a_{ij} y_j \geq 0 \\ & e^T y = 1 \\ & y \geq 0 \end{aligned} \qquad (2)$$

Note that (1) and (2) are duals of each other, so they have the same optimal value. $\square$

The quantity $v = \max_{\substack{x \geq 0 \\ e^T x = 1}} \min_{\substack{y \geq 0 \\ e^T y = 1}} x^T A y$ is the VALUE of the game.

A symmetric game like Rock/Paper/Scissors has value 0, since the roles of the players are reversible.

What about nickle/dime?

$$
\begin{aligned}
v = \max \quad & z \\
\text{s.t.} \quad & z - \sum_i x_i a_{ij} \leq 0 \quad \forall j \\
& e^T x = 1 \\
& x \geq 0
\end{aligned}
$$

$$
\begin{aligned}
= \max \quad & z \\
\text{s.t.} \quad & z - 10x_1 + 15x_2 \leq 0 \\
& z + 15x_1 - 20x_2 \leq 0 \qquad (1) \\
& x_1 + x_2 = 1 \\
& x \geq 0.
\end{aligned}
$$

$$
\begin{aligned}
= \min \quad & w \\
\text{s.t.} \quad & w - 10y_1 + 15y_2 \geq 0 \\
& w + 15y_1 - 20y_2 \geq 0 \qquad (2) \\
& y_1 + y_2 = 1 \\
& y \geq 0
\end{aligned}
$$

Get $v = \frac{-5}{12}$. Achieved by $x_1 = \frac{7}{12}$, $x_2 = \frac{5}{12}$, and by $y_1 = \frac{7}{12}$, $y_2 = \frac{5}{12}$.

So game is advantageous to Player 2,

Variant with two coins of value a and b:
Players (1) and (2)'s LPs become

$$\begin{array}{lrcl}
\max & z & & \\
\text{s.t.} & z - 2ax_1 + (a+b)x_2 & \le & 0 \\
& z + (a+b)x_1 - 2b\,x_2 & \le & 0 \\
& x_1 + x_2 & = & 1 \\
& x & \ge & 0
\end{array} \qquad (1)$$

and

$$\begin{array}{lrcl}
\min & w & & \\
\text{s.t.} & w - 2ay_1 + (a+b)y_2 & \ge & 0 \\
& w + (a+b)y_1 - 2b\,y_2 & \ge & 0 \\
& y_1 + y_2 & = & 1 \\
& y & \ge & 0
\end{array} \qquad (2)$$

The inequalities are active at the optimal soln (else, by C.S., one of the players would have a pure strategy).

So $x_1^* = \dfrac{a+3b}{4a+4b} = y_1^*$ $\qquad x_2^* = \dfrac{3a+b}{4a+4b} = y_2^*$

and $z = w = \dfrac{a^2+3ab}{2a+2b} - \dfrac{3a+b}{4}$ $\qquad \left(= \frac{-5}{12} \text{ if } a=5,\ b=10\ \checkmark\right).$

$$= \frac{1}{4a+4b}\left(2a^2+6ab-3a^2-3ab-ab-b^2\right)$$

$$= \frac{-1}{4a+4b}\left(a^2-2ab+b^2\right)$$

$$= -\frac{(a-b)^2}{4(a+b)} \qquad \left(= \frac{-5}{12} \text{ if } a=5\ b=10\right).$$

So always favours Player 2, if coins have different denominations.

An example when bluffing and underbidding are part of the optimal strategy.

(Chvátal, p. 235)

A deck of cards with three cards: 1, 2, 3.

Each player gets one card. Each player pays an ante of one unit.

Player one can PASS or BET.

A BET requires investing an extra unit.

Player 2 can CALL if player 1 bet: requires an extra unit.
Player 2 can PASS or BET if player 1 passed: betting requires one unit.

If Player 1 passed and player 2 bet, then player 1 can either PASS or CALL.

PLAYER 1		PLAYER 2		PLAYER 1	PAYOFF
BET	$\longrightarrow$	CALL			2 units to holder of higher card
BET	$\longrightarrow$	PASS			1 unit to Player 1
PASS	$\longrightarrow$	PASS			1 unit to holder of higher card
PASS	$\longrightarrow$	BET	$\longrightarrow$	PASS	1 unit to Player 2
PASS	$\longrightarrow$	BET	$\longrightarrow$	CALL	2 units to holder of higher card.

Each player has a mixed strategy.
It is advantageous for Player 1 to occasionally BET when holding a 1
and to occasionally PASS when holding a 3:
this reduces the amount of information available to Player 2 regarding Player 1's card.

Once the optimal strategy is determined, it doesn't matter whether the opponent knows the strategy: it's still just as effective.

Player 1's optimal strategy guarantees a payoff of at least V, regardless of Player 2's strategy.

The value of the inner subproblem cannot be improved, even if x is known

$$\min_{\substack{y \ge 0 \\ e^T y = 1}} x^T A y$$

Non zero sum games

Eg: Prisoners Dilemma:

		Player 2	
		Cooperate	Not
Player 1	Cooperate	(8, 8)	(1, 10)
	Not	(10, 1)	(2, 2)

No stable equilibrium.

Nash equilibrium:

A set of mixed strategies for the players, where no player can gain by unilaterally changing strategy.