PROBABILISTIC ANALYSIS OF SIMPLEX (BORGWARDT, SPRINGER-VERLAG, 1987)

(HAIMOVICH, 1983).

How many pivots are required by simplex 'on average'?

Need to come up with a distribution for instances.

Look at standard dual pair:

$$\begin{array}{llll} \min & c^T x & & \\ \text{s.t.} & Ax = b & (P) & \\ & x \geq 0 & & \end{array} \qquad \begin{array}{lll} \max & b^T y & \\ \text{s.t.} & A^T y \leq c & (D) \end{array}$$

A is $m \times n$, $m < n$.
Assume ~~nondegeneracy~~.

Haimovich : sign invariance:

The distribution of instances is ~~independent of the~~ symmetrical under sign inversions of columns of $\begin{bmatrix} c^T \\ A \end{bmatrix}$.

So, a dual constraint $y_1 + 3y_2 \leq 5$

is as likely as $-y_1 - 3y_2 \leq -5$.

Such a flip makes the opposite halfspace feasible.

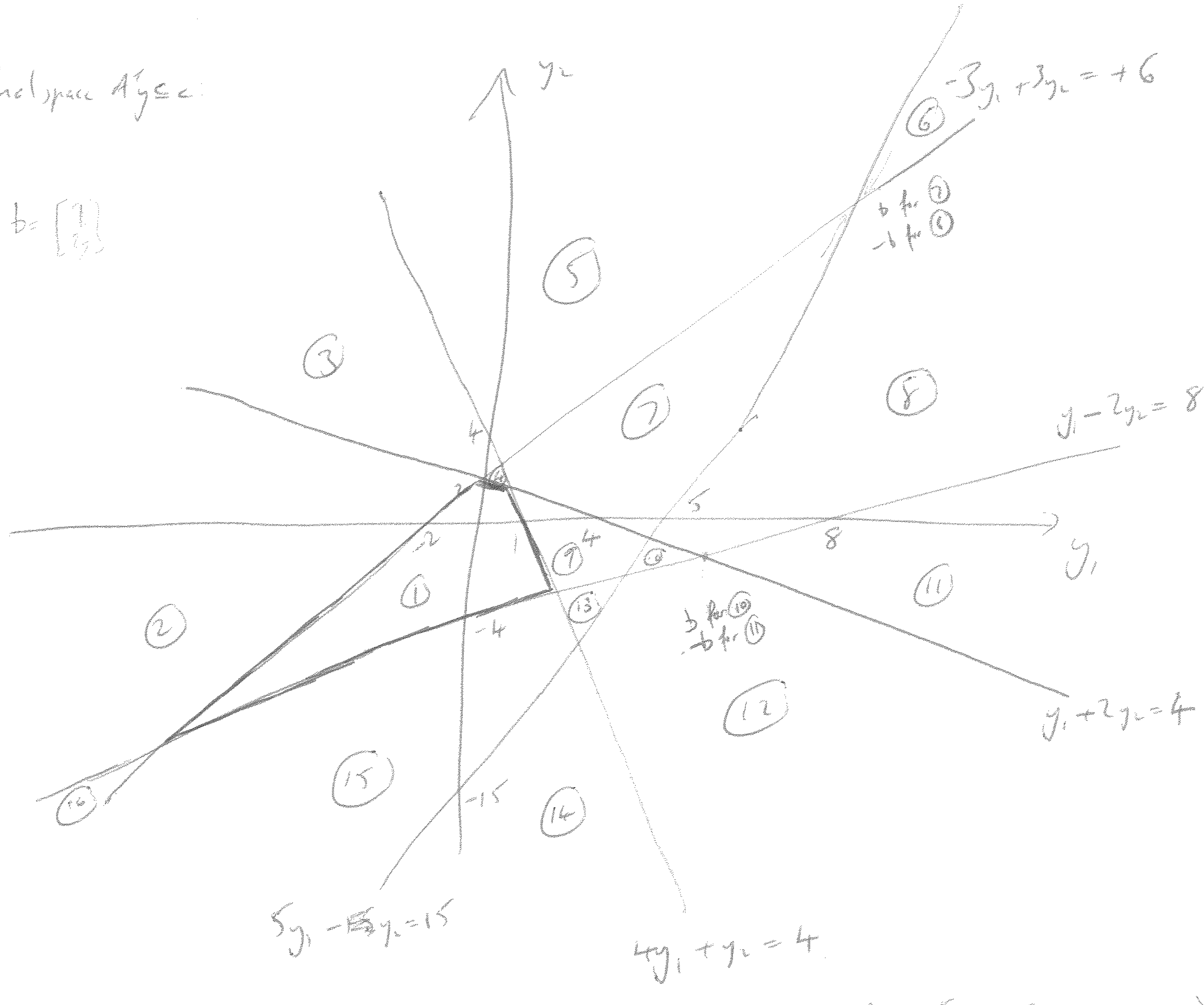
Also assume the distribution is symmetric under changes in the sign of the dual objective vector b.

Eg: $A = \begin{bmatrix} 2 & 1 & -3 & 4 & 5 \\ 1 & -2 & +3 & 1 & -1 \end{bmatrix}$

$c^T = \begin{bmatrix} 4 & 8 & +6 & +4 & 15 \end{bmatrix}$

Dual space $A^T y \le c$:

$b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



$2^5 = 32$ sign choices.

About 16 feasible cells here.

$\binom{5}{2} = 10$ basic ~~feasible~~ solutions.

(Aside: Euler: $F + V = E + 2$.

If join up endpoints to create outer face:

get $E = 25 + 10$

$F = 16 + 1$

$V = 10 + 10$ ✓)

The probabilistic analysis focuses on the dual formulation, with the inequality constraints.

So look at dual vertices and dual pivot sequences.

To find average number of pivots, take:

$$\frac{\text{total \# pivots over all these LPs for a given } (A^T, c)}{\text{\# LPs for a given } (A^T, c)}.$$

#pivots

Notice that each line contains 4 extreme points in our example.

In general, if $m-1$ dual constraints hold at equality, get a line, and this line contains $n-m+1$ extreme points.

(One for each of the other constraints holding at equality.)

line segments is thus $n-m+2$ on each line.

A pivot rule can be constructed so that each line segment is only traversed once, in the whole set of problems. (This requires starting each LP with the solution to another LP, with the same feasible region and with objective $\alpha^T y$. The same α is used for all LPs.)

So total # pivots for a given (A^T, c) under various sign choices:

$$\approx \binom{n}{m-1}(n-m+2).$$

Note that

$$\binom{n}{m-1}(n-m+1) = \binom{n}{m} m.$$

LPs

By the nondegeneracy assumption, each LP has a different optimal solution, for a fixed b.

So the number of LPs for a given b is $\leq \binom{n}{m}$.

Including the objective ~~b~~ also doubles this to $2\binom{n}{m}$.

Use this estimate for # problems.

So: under the sign invariance model, the expected # pivots for the simplex method is

$$\frac{m}{2} = \frac{\binom{n}{m} m}{2\binom{n}{m}}.$$

This agrees well with practical experience.

Typically, for (P), # pivots is linear in m.

E.g., for our A, c:

Try $b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

It seems that each point can be attained as an optimal solution for some LP, with both b and $-b$.

But: perhaps this model produces too many LPs with just a few extreme points.
Also: number of generated problems $\gg$ number of feasible problems (although these numbers are not used in the analysis).

The <u>Rotational Model</u> emphasizes creating feasible LPs.

$$\begin{array}{ll} \max & b^T y \\ \text{s.t.} & A^T y \leq e \end{array} \quad (D) \qquad e = \text{vector of ones.}$$

~~Entries in b and A are distributed on~~

The m-vector b and the n column vectors of A are assumed to be distributed on $\mathbb{R}^m \setminus \{0\}$:

- independently
- identically
- symmetrically under rotations.

Note that $y = 0$ is feasible.

Final results

(0.5.14) on page 30, or (0.11.1) on page 56. for Phase II

(0.5.15) on page 31, or (0.11.2) on page 56 for Phase I + Phase II.

(Note: m and n are reversed from my presentation.)

Better result:

Borgwardt, Math OR, 24(3), 1999, pp 544-603.

The expected number of vertices S ~~#~~ in a path is

$$E_{m,n}(S) = O\left(m^2 n^{1/(m-1)}\right)$$

This is - quadratic in the smaller dimension

- sublinear in the larger dimension

For fixed m, this grows very slowly in n.

The Shadow Vertex Method

Chapter 1, pages 62 →

Concentrate on Phase II, so assume have vertex y^0.

Choose objective function u so that y^0 solves

$$\max \; u^T y \quad \text{s.t.} \; A^T y \leq c.$$

Eg: pick u to be a nonnegative linear combination of the active constraints at y^0.

The shadow vertex algorithm uses the objective $u + \lambda b$,

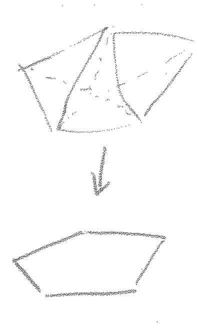
with λ gradually increasing.

Initially $\lambda = 0$.

Every vertex visited is an optimal solution for the objective $u + \lambda b$ for some λ.

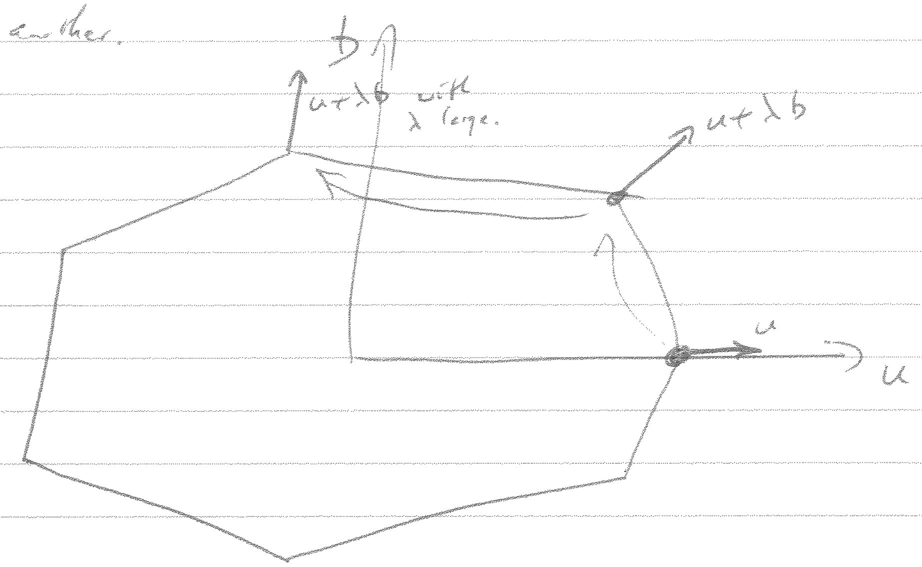
Why "shadow" vertex?

Project the feasible region onto the plane spanned by u and b.



Vertices that map to extreme points in the projection are <u>shadow vertices</u>.

At each iteration, switch on moving from one simplex vertex to another.



Conjectures

Page 61.

Results are for shadow vertex algorithm.
What about other variants of simplex?

The result is an upper bound. How tight is it?

(Tight asymptotically for fixed n as $m \to \infty$.)

What about Phase I? Stated results with Phase I are not so good, with an extra factor of n.